SIMPLE EXPLICIT EXPRESSIONS FOR CALCULATION OF THE HEISLER-GROBER CHARTS

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Abstract

Simple single term approximations for the Heisler cooling charts and the Grober fractional energy loss are presented for the plate, infinite circular cylinder and sphere. These solutions are accurate to within 1% of the series solutions provided the dimensionless time Fo is greater than some critical value Fo_c which lies in the range: $0.18 - 0.24$. Simple explicit expressions are provided for the accurate calculation of the first eigenvalue for all values of the Biot number. Polynomial expressions are presented for the accurate calculation of the roots of the Bessel functions of the first kind of orders zero and one. Expressions are developed for the accurate computation of the Bessel functions of the first kind of orders zero and one. Simple accurate solutions are proposed for calculating the dimensionless temperature and the heat loss fraction for finite circular cylinders, rectangular parallelopipeds and infinite rectangular bars. *Maple V R3* worksheets are given for the accurate calculation of the dimensionless temperature and dimensionless heat loss for the infinite plate, infinite circular cylinder and the sphere.

Nomenclature

A_n	=	Fourier coefficients for dimensionless temperature
B_n	=	Fourier coefficients for heat loss fraction
Bi	=	Biot number; $Bi = h\mathcal{L}/k$
C_1, C_2	=	correlation coefficients
c_p	=	specific heat at constant pressure; $J/kg{\cdot}K$
D	=	diameter of cylinder or sphere; m
Fo	=	Fourier number; $Fo = \alpha t/\mathcal{L}^2$
h	=	heat transfer coefficient; $W/m^2{\cdot}K$
$J_0(\cdot), J_1(\cdot)$	=	Bessel functions, first kind orders 0 and 1
$\mathcal{L}$	=	some characteristic body dimension
N	=	number of panels in trapezoidal approximation
n	=	constant power
Q	=	energy loss; J
Q_i	=	initial internal energy, $\rho c_p V \theta_i$; J
R	=	radius of circular cylinder
S	=	total active surface area; m^2
S_c, S_p, S_s	=	spatial functions for cylinder, plate and sphere
$T(\zeta, Fo)$	=	temperature of body; K
T_f	=	fluid temperature; K
t	=	time; s
V	=	total volume; m^3
X, Y, Z	=	half-dimensions of cuboid and rectangular bar; m
x, y, z	=	Cartesian coordinates

Greek Symbols

α	=	thermal diffusivity; $k/\rho c_p$; m^2/s
β_0, β_1	=	parameters in modified Stokes approximation
δ_n	=	nth root of characteristic equations
$\delta_{n,c}$	=	nth root for infinite cylinder
$\delta_{n,p}$	=	nth root for infinite plate
$\delta_{n,s}$	=	nth root for sphere
δ_1	=	1st root for $0 < Bi < \infty$
$\delta_{1,0}$	=	1st root for $Bi \to 0$
$\delta_{1,\infty}$	=	1st root for $Bi \to \infty$
ϕ	=	dimensionless temperature; θ/θ_i
θ	=	temperature excess; $\theta = T(\zeta, Fo) - T_f$; K
θ_i	=	initial temperature excess, $\theta_i = T(\zeta, 0) - T_f$; K
ρ	=	mass density; kg/m^3
ζ	=	dimensionless position within any body

Subscripts

c	=	infinite cylinder
cp	=	finite cylinder
i	=	initial value

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p	=	infinite plate
x, y, z	=	infinite plates along $x-, y-, z-$coordinates
xy	=	infinite rectangular bar
xyz	=	cuboid
0	=	at very small time
1	=	the first root or first eigenvalue
∞	=	at very large time

Introduction

One-dimensional transient conduction solutions inside plates, infinite circular cylinders and spheres are presented in all heat transfer texts. The governing equations for the three classical geometries are given in Cartesian, circular cylinder and spherical coordinates. The initial and boundary conditions are specified. The dimensionless temperature history (ϕ) which is a function of three dimensionless parameters: position (ζ), time (Fo) and boundary condition (Bi) is presented in symbolic form for the three geometries. All texts present the respective solutions in graphical form, frequently called the Heisler[1] cooling charts (for the temperature at the centerline (plate) or the origin (cylinder, sphere) as a function of Fo and Bi. Auxiliary charts are available for all off-center or off-origin points $0 \leq \zeta < 1$. In addition Heisler charts are presented for small time $Fo < 0.2$.

Grober et al.[2] introduced the charts for the total heat loss fraction Q/Q_i for the three geometries. These charts are presented in great detail in Liukov[3], Grigull and Sandner[4], and all heat transfer texts.

Heisler[1], Liukov[3] and Grigull and Sandner[4] discuss the fact that the temperature and heat loss fraction charts can be computed with acceptable accuracy using the leading term in the respective series solutions. The leading term can be used for all values of Bi provided $Fo \geq Fo_c$ where according to Heisler[1], the critical Fourier number is approximately equal to $Fo_c = 0.24, 0.21, 0.18$ for the infinite plate, infinite circular cylinder and sphere respectively. For $Fo < Fo_c$ more terms in the series solutions are required to give acceptable accuracy. Heisler[1] also noted that more than 80% of the total cooling time is accounted for with the single term solution.

Liukov[3] reports an early attempt to provide approximations for the calculation of the first roots (eigenvalues) of the characteristic equations for the three geometries. He showed graphically that when $\ln(\delta_{1,\infty}/\delta_1 - 1)$ is plotted against $\ln(Bi)$, over the range: $-3 \geq \ln(Bi) \leq 3$, that the points were close to a straight line. A log-linear fit of the data for the three geometries leads to the following correlation equation:

$$\frac{\delta_1}{\delta_{1,\infty}} = \frac{1}{\sqrt{1 + C_1/Bi^{C_2}}} \tag{1}$$

where $\delta_{1,\infty}$ is the value of δ_1 at $Bi = \infty$. The values of $\delta_{1,\infty}$ are: $\pi/2, 2.4048255.., \pi$ for the infinite plate, the infinite circular cylinder and the sphere respectively. Liukov[3] reports the correlation coefficients for the plate: $C_1 = 2.24, C_2 = 1.02$, the cylinder: $C_1 = 2.45, C_2 = 1.04$, and the sphere: $C_1 = 2.70, C_2 = 1.07$. This correlation equation for the first eigenvalues gives acceptable values for $Bi \geq 100$; otherwise the errors are much greater than 1%. The largest errors occur when $Bi \leq 0.1$. It is unknown what errors are introduced into the computation of the Fourier coefficients A_1 and B_1 by the use of this correlation equation.

Chen and Kuo[5] applied the heat balance integral method to obtain approximate solutions for the infinite plate and the infinite circular cylinder. These equations which are reported in Chapman[6] can be evaluated by means of programmable calculators, and they are said to be accurate provided $Fo > Fo_c$. Since these equations are lengthy and involved, they will not be presented here.

Some of the recently published heat transfer texts: Chapman[6] , Bejan[7] , Holman[8] , Incropera and DeWitt[9] , Mills[10] and White[11] recognize that the series solutions converge to the leading term for *long* times, i.e. $Fo > 0.2$ with errors of about 1 %. They present tables for the roots (eigenvalues) of the corresponding characteristic equations and the Fourier coefficients: A_1 and B_1 that appear in the dimensionless temperature and heat loss fraction expressions. For values of Bi not given in the tables, it is necessary to employ interpolation methods to use the tabulated values. Once the eigenvalues are known, the evaluation of the Bessel functions $J_0(\cdot)$ and $J_1(\cdot)$ that appear in the characteristic equation for the circular cylinder, and the Fourier-Bessel coefficients that appear in the temperature and heat loss expressions must be considered. These calculations are tedious and prone to errors, and unnecessary with the availability of programmable calculators and computers.

There is, therefore, a need to develop simple, accurate equations for the computation of the first root of the characteristic equations for the three geometries. Secondly, there is a need to develop simple relationships for the accurate calculation of the Bessel functions that appear in the solutions for the infinite circular cylinder. These relationships will then be used to develop simple accurate relationships for the accurate calculation of the dimensionless temperature and the heat loss fraction for the three geometries for $Fo \geq Fo_c$. Finally, by means of superposition, expressions will be developed for calculating the dimensionless temperature within composite bodies such as cuboids (Fig. 1), infinite rectangular bars (Fig. 2), and finite circular cylinders (Fig. 3). Also, by means of the Langston[12] relationships for the determination of the heat loss fraction for composite bodies, a method will be proposed for the simple, but accurate, calculation of the heat loss fraction from: cuboids, infinite rectangular bars, infinite plates, finite and infinitely long circular cylinders.

Heisler Dimensionless Temperature Charts

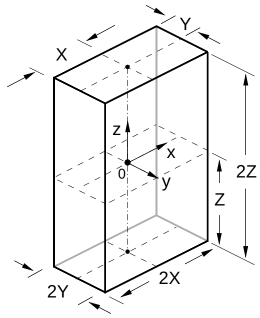

Fig. 1 The cuboid

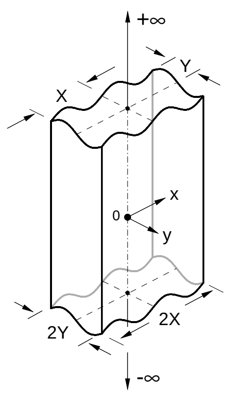

Fig. 2 The infinite rectangular bar

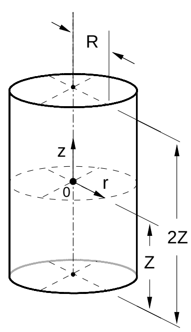

Fig. 3 The finite circular cylinder

The Heisler[1] dimensionless temperature charts that appear in all heat transfer texts were developed for the classic geometries: plate, infinite circular cylinder and sphere. Since the dimensionless temperature ϕ depends on three dimensionless parameters: Bi, Fo, ζ where Bi is the Biot number, Fo is the Fourier number, and ζ is the dimensionless position, it is not possible to show the temperature at any point within the solid at any arbitrary time.

The dimensionless temperature charts are based on the general solution:

$$\phi = \sum_{n=1}^{\infty} A_n \exp\left(-\delta_n^2 Fo\right) S\left(\delta_n \zeta\right) \tag{2}$$

where A_n are the temperature Fourier coefficients that are functions of the boundary condition through Bi and the initial condition, δ_n are the eigenvalues which are the positive roots of the characteristic equation, and $S\left(\delta_n \zeta\right)$ is the position function. For the three geometries the position function has the forms:

Plate

$$S_p = \cos\left(\delta_n \zeta\right) \tag{3}$$

Circular Cylinder

$$S_c = J_0\left(\delta_n \zeta\right) \tag{4}$$

Sphere

$$S_s = \frac{\sin\left(\delta_n \zeta\right)}{\left(\delta_n \zeta\right)} \tag{5}$$

Fourier Coefficients for Temperature

The Fourier coefficients for temperature for the three geometries have the following forms:

Plate

$$A_{n,p} = \frac{2 \sin \delta_n}{\delta_n + \sin \delta_n \cos \delta_n} \tag{6}$$

or

$$A_{n,p} = (-1)^{n+1} \frac{2Bi\left(Bi^2 + \delta_n^2\right)^{1/2}}{\delta_n\left(Bi^2 + Bi + \delta_n^2\right)} \tag{7}$$

Circular Cylinder

$$A_{n,c} = \frac{2J_1\left(\delta_n\right)}{\delta_n\left[J_0^2\left(\delta_n\right) + J_1^2\left(\delta_n\right)\right]} \tag{8}$$

or

$$A_{n,c} = \frac{2Bi}{J_0\left(\delta_n\right)\left[\delta_n^2 + Bi^2\right]} = \frac{2}{\delta_n\left[1 + \delta_n^2/Bi^2\right] J_1\left(\delta_n\right)} \tag{9}$$

Sphere

$$A_{n,s} = \frac{2\left(\sin\delta_n - \delta_n\cos\delta_n\right)}{\delta_n - \sin\delta_n\cos\delta_n} \quad (10)$$

$$A_{n,s} = (-1)^{n+1}\frac{2Bi\left[\delta_n^2 + (Bi-1)^2\right]^{1/2}}{\left(\delta_n^2 + Bi^2 - Bi\right)} \quad (11)$$

The eigenvalues that appear in the above relationships are the positive roots of the characteristic equations which have the following forms for the three geometries:

Plate

$$x\sin x = Bi\cos x \quad (12)$$

Circular Cylinder

$$xJ_1(x) = BiJ_0(x) \quad (13)$$

Sphere

$$(1-Bi)\sin x = x\cos x \quad (14)$$

Analysis of the Solutions

$\mathbf{Bi \to 0}$

For these three geometries for all dimensionlesss time $Fo > 0$, as $Bi \to 0$, the first Fourier coefficient $A_1 \to 1$ and all other Fourier coefficients $A_n \to 0$ for $n \geq 2$. The first root of the three characteristic equations approaches zero in the following manner:

Plate

$$\delta_{1,p} \to \sqrt{Bi} \quad (15)$$

Circular Cylinder

$$\delta_{1,c} \to \sqrt{2Bi} \quad (16)$$

Sphere

$$\delta_{1,s} \to \sqrt{3Bi} \quad (17)$$

and the respective dimensionless temperature solutions become:

Plate

$$\phi_p = e^{-BiFo} \quad (18)$$

Circular Cylinder

$$\phi_c = e^{-2BiFo} \quad (19)$$

Sphere

$$\phi_s = e^{-3BiFo} \quad (20)$$

which are seen to be particular cases of the general lumped parameter solution:

$$\phi = e^{-\dfrac{hS}{\rho c_p V}t} \quad (21)$$

where S is the total active heat transfer surface of the geometry and V is its volume.

$\mathbf{Bi \to 0}$ Limit

At this limit the eigenvalues for the three geometries go to the following relationships:

Plate

$$\delta_{n,p} = (n-1)\frac{\pi}{2}, \qquad n \geq 2 \quad (22)$$

Circular Cylinder

$$\delta_{n,c} = \frac{\beta_1}{4}\left[1 - \frac{6}{\beta_1^2} - \frac{6}{3\beta_1^4} - \frac{2358}{5\beta_1^6}\right], \qquad n = 1,2,3... \quad (23)$$

with $\beta_1 = \pi(4n+1)$. The above relationship is a modification of the Stokes approximation (Abramowitz and Stegun[13]). It gives acceptable values of the roots of $J_1(\cdot) = 0$ for all values $n \geq 1$. The largest error is approximately 0.0015% when $n = 1$ and the error is much smaller for all $n \geq 2$.

Sphere

There is no simple relationship for the eigenvalues $\delta_{n,p}$ for $n \geq 2$. The eigenvalues are the roots of

$$x\cos(x) - sin(x) = 0 \quad (24)$$

Numerical methods are required to find the roots which lie in the interval: $(n-1)\pi, (2n-1)\pi/2$. The first nine roots are approximately:

$$\begin{aligned}
x_1 &= 0\\
x_2 &= 4.493409\\
x_3 &= 7.725252\\
x_4 &= 10.904122\\
x_5 &= 14.066194\\
x_6 &= 17.220755\\
x_7 &= 20.371303\\
x_8 &= 23.519453\\
x_9 &= 26.666054
\end{aligned}$$

For very large values of n the roots approach the value $x_n \to (2n-1)\pi/2$.

$\mathbf{Bi \to \infty}$ Limit

At this limit, the roots (eigenvalues) of the characteristic equations are given by the following relationships:

Plate

$$\delta_n = (2n-1)\frac{\pi}{2} \qquad n = 1,2,3... \quad (25)$$

Circular Cylinder

$$\delta_n = \frac{\beta_0}{4}\left[1 + \frac{2}{\beta_0^2} - \frac{62}{3\beta_0^4} + \frac{7558}{15\beta_0^6}\right], \qquad n = 1,2,3... \quad (26)$$

with $\beta_0 = \pi(4n-1)$. The above relationship is a modification of the Stokes approximation (Abramowitz and Stegun[13]). It gives acceptable values of the roots of

$J_0(\cdot) = 0$ for all values $n \geq 1$. The largest error is approximately -0.0024% when $n = 1$, and the error is much smaller for all $n \geq 2$.

Sphere

$$\delta_n = n\pi \qquad n = 1, 2, 3, ... \tag{27}$$

The Fourier coefficients for temperature are determined by means of the following expressions:

Plate

$$A_n = (-1)^{n+1} \frac{4}{(2n-1)\pi} \qquad n = 1, 2, 3... \tag{28}$$

Circular Cylinder

$$A_n = \frac{2}{\delta_n J_1(\delta_n)} \qquad n = 1, 2, 3... \tag{29}$$

Sphere

$$A_n = 2(-1)^{n+1} \qquad n = 1, 2, 3... \tag{30}$$

Heat Loss Fraction Charts

The heat loss fraction Q/Q_i where $Q_i = \rho c_p V \theta_i$ is the initial total internal energy depends on the boundary condition parameter Bi and the dimensionless time Fo in the following way:

$$\frac{Q}{Q_i} = 1 - \sum_{n=1}^{\infty} B_n \exp\left(-\delta_n^2 Fo\right) \tag{31}$$

The Fourier coefficients B_n are given by the following relationships for the three geometries:

Plate

$$B_{n,p} = A_{n,p} \frac{\sin \delta_n}{\delta_n} = \frac{2Bi^2}{\delta_n^2 \left(Bi^2 + Bi + \delta_n^2\right)} \tag{32}$$

Circular Cylinder

$$B_{n,c} = 2A_{n,c} \frac{J_1(\delta_n)}{\delta_n} = \frac{4Bi^2}{\delta_n^2 \left(\delta_n^2 + Bi^2\right)} \tag{33}$$

Sphere

$$B_{n,s} = \frac{6Bi^2}{\delta_n^2 \left(\delta_n^2 + Bi^2 - Bi\right)} \tag{34}$$

Analysis of the Heat Loss Coefficients

The heat loss coeffcients B_n have particular values in the two limits: $Bi \to 0$ and $Bi \to \infty$. In the first limit, all Fourier coeffcients for heat loss fraction are equal to zero for $n \geq 2$.

In the second limit they are given by:

Plate

$$B_{n,p} = \frac{8}{\pi^2 (2n-1)^2} \qquad n = 1, 2, 3... \tag{35}$$

Circular Cylinder

$$B_{n,c} = \frac{4}{\delta_n^2} \qquad n = 1, 2, 3... \tag{36}$$

where δ_n are the roots of $J_0(\delta_n) = 0$ which are given above.

Sphere

$$B_{n,s} = \frac{6}{(n\pi)^2} \qquad n = 1, 2, 3... \tag{37}$$

Clearly the heat loss coefficients are easily computed as $Bi \to \infty$ and therefore the heat loss fraction can be determined without difficulty.

Numerical Solutions

Accurate numerical results for the three geometries can be obtained easily by means of a Computer Algebra System such as *Maple*[14], *MathCAD*[15], *Mathematica*[16] or *MATLAB*[17]. The proposed solutions and procedure can also be implemented in spreadsheets.

Maple[14] V R3 worksheets for the plate, infinite circular cylinder and the sphere are presented in the Appendix. For each geometry the input parameters are: Bi, Fo, ζ, N where N is the number of terms in the partial sum. It is recommended that $N = 5$ although it can be set to any integer value $N > 1$. In each worksheet, the first five inputs are: i) the definition of the characteristic equation, ii) definition of the Fourier coefficient for temperature, iii) definition of the Fourier coefficient for heat loss fraction, iv) definition of the dimensionless temperature, and v) definition of the heat loss fraction. The next five inputs create lists for the N: i) eigenvalues, ii) coefficients A_n, iii) coefficients B_n, iv) dimensionless temperature terms ϕ_n, and v) heat loss fraction terms $Q/Q_{i,n}$. The last two inputs give the dimensionless temperature and the heat loss fraction for the given values of Bi, Fo and ζ.

Approximations of Bessel Functions

There are several methods that can be used to compute the Bessel functions: $J_0(x)$ and $J_1(x)$. There are polynomial approximations (Abramowitz and Stegun[13]) available for all positive values of x for both Bessel functions. These polynomial approximations are based on many terms that require space, and they are somewhat difficult to implement in spreadsheets or in a programmable calculator. The following expression which is based on the application of the trapezoidal rule to the integral form of $J_\nu(x)$ for arbitrary order was developed by means of the *Maple* function *trapezoid* which is found in the student package:

$$J_\nu(x) = \frac{1}{2N} + \frac{\cos(\nu\pi)}{2N} + \frac{1}{N} \sum_{i=1}^{N-1} \cos\left(x \sin\left(\frac{i\pi}{N}\right) - \frac{\nu i \pi}{N}\right) \tag{38}$$

where N is the number of panels and ν is the order of the Bessel function. From this general expression one can develop expressions for both $J_0(x)$ and $J_1(x)$ for different ranges of x and for different accuracy. For the range: $0 \leq x \leq 2.4048$ that is applicable for the first term and for the entire range of Bi, one can use 6 panels in the above expression to obtain the following expressions which provide accurate values of the two Bessel functions:

$$J_0(x) = \frac{1}{6}\left[1 + \sum_{i=1}^{5} \cos\left(x \sin\left(\frac{1}{6} i\pi\right)\right)\right] \tag{39}$$

and

$$J_1(x) = \frac{1}{6}\left[\sum_{i=1}^{5} \cos\left(x \sin\left(\frac{1}{6} i\pi\right) - \frac{1}{6} i\pi\right)\right] \tag{40}$$

Explicit Solutions of Characteristic Equations

The observations regarding the relationship of the first root δ_1 of the the three characteristic equations with respect to Bi reported by Liukov[3] and the results of the analysis of the theoretical solutions presented above suggests that it is possible to use the asymptotic values: $\delta_{1,0}$ and $\delta_{1,\infty}$ corresponding to $Bi \to 0$ and $Bi \to \infty$ respectively to develop interpolation functions for δ_1 which will be accurate for all values of Bi. Plotting the ratio $\delta_{1,\infty}/\delta_1$ versus Bi gives a function that various smoothly between the asymptote: $\delta_{1,\infty}/\delta_{1,0}$ for $Bi \to 0$ and the other asymptote is equal to 1 for $Bi \to \infty$.

Based on the above observations the explicit solutions (first eigenvalues) of the characteristic equations for the three geometries can be written in the general form:

$$\delta_1 = \frac{\delta_{1,\infty}}{\left[1 + \left(\frac{\delta_{1,\infty}}{\delta_{1,0}}\right)^n\right]^{1/n}} \tag{41}$$

The form is based on the method first proposed by Churchill and Usagi[14]. The approximate explicit solution always gives very accurate values for very small and very large values of Bi independent of the value of the parameter n. To obtain accurate values for intermediate values of Bi: $(0.5 \leq Bi \leq 5)$ it is necessary to find appropriate values of n. The values: $(n = 2.139, n = 2.238, n = 2.314)$ for the plate, cylinder and sphere respectively, provide values of δ_1 which differ by less than 0.4 % from the exact values of δ_1. This accuracy is acceptable for most applications. To develop more accurate solutions for the intermediate range it may be necessary to find relationships between the parameter n and Bi for each geometry.

Temperature and Heat Loss Fraction for Composite Geometries

The basic solutions given above can be used to develop solutions for composite geometries such as cuboids and finite circular cylinders with convection cooling at all boundary surfaces. Since the cuboid solution is based on the superposition of three plate solutions, it reduces to the infinite rectangular bar solution and the infinite plate solution.

The dimensionless temperature and heat loss fraction solutions for the cuboid and the finite circular cylinder will be presented in the following sections.

For the general case of a cuboid (Fig 1): $-X \leq x \leq X, -Y \leq y \leq Y, -Z \leq z \leq Z$ that is cooled at its perpendicular faces: $x = \pm X, y = \pm Y$, $z = \pm Z$, through uniform heat transfer coefficients: h_x, h_y, h_z, there are three Biot numbers to consider: Bi_x, Bi_y, Bi_z. The cooling of a cuboid is also characterized by three Fourier numbers: Fo_x, Fo_y, Fo_z.

Dimensionless Temperature for Cuboids

The dimensionless temperature at any point within the cuboid for arbitrary time can be obtained by the means of the product of the solutions for three infinite plates:

$$\phi_{xyz}(x, y, z, t) = \phi_{x,p}\phi_{y,p}\phi_{z,p} \tag{42}$$

where the basic infinite plate solution given by Eq. (2) is used three times:

$$\phi_{x,p} = A_{1,x} \exp\left(-\delta_{1,x}^2 Fo_x\right) \cos\left(\delta_{1,x} x/X\right) \tag{43}$$

$$\phi_{y,p} = A_{1,y} \exp\left(-\delta_{1,y}^2 Fo_y\right) \cos\left(\delta_{1,y} y/Y\right) \tag{44}$$

$$\phi_{z,p} = A_{1,z} \exp\left(-\delta_{1,z}^2 Fo_z\right) \cos\left(\delta_{1,z} z/Z\right) \tag{45}$$

The corresponding eigenvalues: $\delta_{1,x}, \delta_{1,y}, \delta_{1,z}$ are dependent on the respective Biot numbers: Bi_x, Bi_y, Bi_z.

The Fourier coefficients: $A_{1,x}, A_{1,y}, A_{1,z}$ are determined according to Eqs. (6) or (7). The eigenvalues: $\delta_{1,x}, \delta_{1,y}, \delta_{1,z}$ are calculated by means of the general explicit relationship developed for δ_1.

Rectangular Plates

The dimensionless temperature and heat loss fraction from rectangular plates or bars (Fig. 2): $-X \leq x \leq X, -Y \leq y \leq Y$ is a special case of the cuboid solution. Here two Biot numbers: Bi_x, Bi_y and two Fourier numbers: Fo_x, Fo_y are required to characterize its cooling.

Dimensionless Temperature of Rectangular Plates

The dimensionless temperature at any point within infinite rectangular plates for arbitrary time can be obtained by the means of the product of the solutions for two infinite plates:

$$\phi_{xy}(x, y, t) = \phi_{x,p}\phi_{y,p} \tag{46}$$

where the basic infinite plate solution given by Eq. (2) is used two times:

$$\phi_{x,p} = A_{1,x} \exp\left(-\delta_{1,x}^2 Fo_x\right) \cos\left(\delta_{1,x} x/X\right) \tag{47}$$

$$\phi_{y,p} = A_{1,y} \exp\left(-\delta_{1,y}^2 Fo_y\right) \cos\left(\delta_{1,y} y/Y\right) \tag{48}$$

Heat Loss Fraction for Cuboids

The heat loss fraction can be determined by means of the relationship developed by Langston[12]:

$$\left(\frac{Q}{Q_i}\right)_{xyz} = \left(\frac{Q}{Q_i}\right)_x + \left(\frac{Q}{Q_i}\right)_y \left[1 - \left(\frac{Q}{Q_i}\right)_x\right] + \left(\frac{Q}{Q_i}\right)_z \left[1 - \left(\frac{Q}{Q_i}\right)_x\right] \left[1 - \left(\frac{Q}{Q_i}\right)_y\right] \tag{49}$$

where $Q_i = \rho c_p 8XYZ\theta_i$.

Heat Loss Fraction from Rectangular Plates

The heat loss fraction from a rectangular plate: $-X \leq x \leq X, -Y \leq y \leq Y$ is a special case of the cuboid solution. Here two Biot numbers: Bi_x, Bi_y and two Fourier numbers: Fo_x, Fo_y are required to characterize its cooling. The heat loss fraction is obtained by following relationship(Langston[12]):

$$\left(\frac{Q}{Q_i}\right)_{xy} = \left(\frac{Q}{Q_i}\right)_x + \left(\frac{Q}{Q_i}\right)_y - \left(\frac{Q}{Q_i}\right)_x \left(\frac{Q}{Q_i}\right)_y \tag{50}$$

where $Q_i = \rho c_p 4XY\theta_i$.

In the above relationships the component heat loss fractions are obtained by means of the following expressions:

$$\left(\frac{Q}{Q_i}\right)_x = 1 - B_{1,x} \exp\left(-\delta_{1,x}^2 Fo_x\right) \tag{51}$$

and

$$\left(\frac{Q}{Q_i}\right)_y = 1 - B_{1,y} \exp\left(-\delta_{1,y}^2 Fo_y\right) \tag{52}$$

and

$$\left(\frac{Q}{Q_i}\right)_z = 1 - B_{1,z} \exp\left(-\delta_{1,z}^2 Fo_z\right) \tag{53}$$

The corresponding eigenvalues: $\delta_{1,x}, \delta_{1,y}, \delta_{1,z}$ are dependent on the respective Biot numbers: Bi_x, Bi_y, Bi_z.

Finite Circular Cylinders

The dimensionless temperature and the heat loss fraction from a finite circular cylinder (Fig. 3): $0 \leq r \leq R, -X \leq x \leq X$ is based on the superposition of the infinite circular cylinder and infinite plate solutions. Here two Biot numbers: $Bi_p = h_x X/k, Bi_c = h_r R/k$ and two Fourier numbers: $Fo_p = \alpha t/X^2, Fo_c = \alpha t/R^2$ are required to characterize its cooling. The heat transfer coefficients are identical over the two ends: $x = \pm X$, but different from the side heat transfer coefficient h_r.

Dimensionless Temperature for Finite Circular Cylinders

The dimensionless temperature at any point within the finite circular cylinder for arbitrary time can be obtained through the product of the solutions for the infinite circular cylinder and the infinite plate:

$$\phi_{cp}(r, z, t) = \phi_c \phi_p \tag{54}$$

where

$$\phi_c = A_{1,c} \exp\left(-\delta_{1,c}^2 Fo_c\right) J_0\left(\delta_{1,c} r/R\right) \tag{55}$$

and

$$\phi_p = A_{1,p} \exp\left(-\delta_{1,p}^2 Fo_p\right) \cos\left(\delta_{1,p} x/X\right) \tag{56}$$

Heat Loss Fraction for Finite Circular Cylinders

The heat loss fraction is obtained by following relationship (Langston[12]):

$$\left(\frac{Q}{Q_i}\right)_{cp} = \left(\frac{Q}{Q_i}\right)_c + \left(\frac{Q}{Q_i}\right)_p - \left(\frac{Q}{Q_i}\right)_c \left(\frac{Q}{Q_i}\right)_p \tag{57}$$

where $Q_i = \rho c_p 2\pi X R^2 \theta_i$.

In the above expressions the component heat loss fractions are obtained by means of the following expressions:

$$\left(\frac{Q}{Q_i}\right)_c = 1 - B_{1,c} \exp\left(-\delta_{1,c}^2 Fo_c\right) \tag{58}$$

and

$$\left(\frac{Q}{Q_i}\right)_p = 1 - B_{1,p} \exp\left(-\delta_{1,p}^2 Fo_p\right) \tag{59}$$

The corresponding eigenvalues: $\delta_{1,c}, \delta_{1,p}$ are dependent on the respective Biot numbers: Bi_c, Bi_p.

Summary

Simple, explicit and accurate expressions were developed for the calculation of the first roots of the characteristic equations for infinite plates, infinite circular cylinders and spheres for all values of the Biot number. Accurate polynomial expressions which are modifications of the Stokes approximations for the roots of the Bessel functions: $J_0(\cdot) = 0$ and $J_1(\cdot) = 0$ are proposed. Simple expressions based on the application of the trapezoidal rule to the integral form of the Bessel function of the first kind and arbitrary order ν are presented. These expressions are expanded in terms of the trigonometric functions which are easily computed in spreadsheets and with programmable calculators. *Maple V R3* worksheets are presented in the appendix for accurate calculation of dimensionless temperature and dimensionless heat loss for the plates, cylinders and spheres for all values of the Biot number and any value of the Fourier number provided $Fo \geq 0.01$. Simple single term expressions are given for the accurate calculation of the dimensionless temperature and the dimensionless heat loss fraction for infinite plates, infinite cylinders and spheres. These single term expressions are used to develop expressions, based on superposition and the method of Langston[12], for the calculation of dimensionless temperature and the dimensionless heat loss fraction of bodies such finite circular cylinders, cuboids and infinite rectangular bars. The proposed expressions are simple and accurate provided $Fo \geq Fo_c$. They should replace the tabular method currently presented in all heat transfer texts. The tabular method effectively replaces the Heisler[1] and Grober[2] charts.

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Appendix
Maple V R3 Worksheets

The *Maple*[14] worksheets for the infinite plates and circular cylinders, and the sphere are presented here. They are valid for any value of ζ and for all values of Bi. The worksheets can handle small dimensionless times: $Fo > 0.01$ by increasing the number of terms in the summation. The input parameters for each worksheet are: Bi, Fo, ζ, N where N is the number of terms in the summation. It is recommended that the number of terms should be limited to $N = 5$ or less for most problems.

```
Maple Worksheet for Plates

restart:
case:= (Bi = 0.3, Fo = 1, zeta = 0, N = 5):
ce:= x*sin(x) - Bi*cos(x) = 0:
A:= 2*sin(x)/(x + sin(x)*cos(x)):
B:= A*sin(x)/x:
phi:= A*exp(- x^2*Fo)*cos(x*zeta):
Q_Qi:= B*exp(- x^2*Fo):

xvals:= [seq(fsolve(subs(case, ce),
x = j*Pi..(j + 1/2)*Pi, j = 0..rhs(case[4]))]:
As:= evalf([seq(subs(x = xvals[j], A),
j = 1..nops(xvals))]):
Bs:= evalf([seq(subs(x = xvals[j], B),
j = 1..nops(xvals))]):
phis:=
```

```
[evalf(seq(subs(case, x = xvals[j], phi),
j = 1..nops(xvals)))]:
Q_Qis:=
[evalf(seq(subs(case, x = xvals[j], Q_Qi),
j = 1..nops(xvals)))]:
plate_temp:=
evalf(convert([seq(phis[j],
j = 1..nops(xvals))], `+`), 4);
plate_heat_loss:=
evalf(1 - convert([seq(Q_Qis[j],
j = 1..nops(xvals))], `+`), 4);
```

Maple Worksheet for Cylinders

```
restart:
case:= (Bi = 0.3, Fo = 1, zeta = 0, N = 5):
ce:= x*BesselJ(1, x) - Bi*BesselJ(0, x) = 0:
A:= 2*BesselJ(1, x)/(x*(BesselJ(0, x)^2 +
BesselJ(1, x)^2)):
B:= A*2*BesselJ(1, x)/x:
phi:= A*exp(- x^2*Fo)*BesselJ(0, x*zeta):
Q_Qi:= B*exp(- x^2*Fo):

xvals:= [seq(fsolve(subs(case, ce),
x = (j - 1)*Pi..j*Pi),
j = 0..rhs(case[4]))]:
As:= evalf([seq(subs(x = xvals[j], A),
j = 1..nops(xvals))]):
Bs:= evalf([seq(subs(x = xvals[j], B),
j = 1..nops(xvals))]):
phis:=
[evalf(seq(subs(case, x = xvals[j], phi),
j = 1..nops(xvals)))]:
Q_Qis:=
[evalf(seq(subs(case, x = xvals[j], Q_Qi),
j = 1..nops(xvals)))]:
cylinder_temp:=
evalf(convert([seq(phis[j],
j = 1..nops(xvals))], `+`), 4);
cylinder_heat_loss:=
evalf(1 - convert([seq(Q_Qis[j],
j = 1..nops(xvals))], `+`), 4);
```

Maple Worksheet for Spheres.

```
#Bi cannot be set to 1 and zeta cannot be set to 0.
#For Bi = 1, put Bi = 1.000001, and for zeta = 0,
#put zeta = 0.000001.

restart:
case:= (Bi = 0.3, Fo = 1, zeta = 0, N = 5):
ce:= x*cos(x) - (1 - Bi)*sin(x) = 0:
A:= 2*(sin(x) - x*cos(x))/(x - sin(x)*cos(x)):
B:= A*3*(sin(x) - x*cos(x))/x^3:
phi:= A*exp(- x^2*Fo)*sin(x*zeta)/(x*zeta):
Q_Qi:= B*exp(- x^2*Fo):

xvals:= [seq(fsolve(subs(case, ce),
x = (j - 1)*Pi..j*Pi),
j = 0..rhs(case[4]))]:
As:= evalf([seq(subs(x = xvals[j], A),
j = 1..nops(xvals))]):
Bs:= evalf([seq(subs(x = xvals[j], B),
j = 1..nops(xvals))]):
phis:= [evalf(seq(subs(case,
x = xvals[j], phi),
j = 1..nops(xvals)))]:
Q_Qis:=
[evalf(seq(subs(case, x = xvals[j], Q_Qi),
j = 1..nops(xvals)))]:
sphere_temp:=
evalf(convert([seq(phis[j],
j = 1..nops(xvals))], `+`), 4);
sphere_heat_loss:=
evalf(1 - convert([seq(Q_Qis[j],
j = 1..nops(xvals))], `+`), 4);
```