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Lumped Capacitance Model With Ohmic Heating

Lumped capacitance model (valid for $Bi < 0.2$) for a system (long constant cross-section wire) which has uniformly distributed heat sources due to ohmic heating, and convective cooling from its surface through a uniform and constant heat transfer coefficient into a fluid at constant temperature.

The system is initial at some arbitrary temperature which different from the surrounding fluid temperature.

Derive the general ordinary differential equation, and then obtain its solution.

Derivation
There are *three* energy components to consider:

1. **Energy generation rate within the system due to ohmic heating**
2. **Energy loss rate (heat transfer rate) from the system boundaries into the surroundings by convective cooling**
3. **Energy storage rate within the system**

The three components can be expressed as:

$$\dot{E}_{\text{gen}} = SV = J^2 \rho_{\text{e}} A_{\text{c}} L$$

$$Q_{\text{loss}} = hA_{\text{s}} \left(T(t) - T_f\right) = hPL \left(T(t) - T_f\right)$$

$$\dot{E}_{\text{storage}} = \rho C_p V \frac{d\left(T(t) - T_f\right)}{dt} = \rho C_p A_{\text{c}} L \frac{d\left(T(t) - T_f\right)}{dt}$$

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where $A_{\text{c}}, A_{\text{s}}, L, P, V$ are the cross-sectional area, surface area, length, perimeter and volume of the system. The thermal and electrical properties of the system are: ρ, the mass density, C_p, the specific heat, ρ_{e}, the electrical resistivity of the system. The other physical parameters are: S, the volumetric heat generation rate, J, the current density, and h, the heat transfer coefficient. The system and fluid temperatures are denoted as $T(t)$ and T_f. For the subsequent analysis it is convenient to introduce the temperature *excess* defined as $\theta(t) = T(t) - T_f$.

Ordinary Differential Equation

The ordinary differential equation is obtained by means of an energy balance over the system boundaries, and division by the volume of the system.

$$\text{ODE} = \frac{\dot{E}_{\text{gen}} - Q_{\text{loss}} - \dot{E}_{\text{storage}}}{V} = 0$$

After substitution and cancellation we get the ordinary differential equation:

$$\rho C_p \frac{d\theta(t)}{dt} + \frac{hP}{A_{\text{c}}}\theta(t) = J^2 \rho_{\text{e}}$$

which is expressed in the conventional form:

$$\frac{d\theta(t)}{dt} + m\ \theta(t) = n$$

with constant physical parameters:

$$m = \frac{hP}{\rho C_P A_{\text{c}}}$$

and

$$n = \frac{J^2 \rho_e}{\rho C_P}$$

The units of the parameters are: $m\,[s^{-1}]$ and $n\,[K s^{-1}]$ to be consistent with the units of the first term.

The first order nonhomgeneous ordinary differential equation requires one initial condition which is

$$t = 0, \qquad \theta(0) = \theta_0 = T_0 - T_f$$

Solutions

The general solution is

$$\theta(t) = \frac{n}{m} + \left(\theta_0 - \frac{n}{m}\right) e^{-m\,t}, \qquad t > 0$$

We note that the units of n/m and m are K and s^{-1} respectively. The characteristic time (time constant) of the system is

$$t_{\rm c} = \frac{1}{m} = \frac{\rho C_P A_{\rm c}}{hP}$$

The steady-state solution corresponds to long times where $t \to \infty$ or whenever $mt >> 0$. The steady-state solution is

$$\theta_{\rm ss} = \frac{n}{m} = \frac{J^2 \rho_{\rm e} A_{\rm c}}{hP}$$

The steady-state solution can be obtained directly from an energy balance based on $\dot{E}_{\rm gen}$ and $Q_{\rm loss}$.