UNIVERSITY OF WATERLOO

Department of Mechanical Engineering

ME 303 Advanced Engineering Mathematics

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Project due date: Friday, July 9 at 9:30 AM

Consider the following linear, second-order, nonhomogeneous PDE:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{S}{k} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \quad t > 0, \quad 0 < x < L$$

where k is the thermal conductivity, α is the thermal diffusivity, and S represents the uniformly distributed volumetric heat sources. The length of the rod is L and its diameter is D. The temperature excess is defined as $\theta(x,t) = T(x,t) - T_\infty$, where T_∞ represents the ambient temperature.

The two boundary conditions (BCs) for $t > 0$ are:

$$x = 0, \quad \frac{\partial \theta(0,t)}{\partial x} = 0, \quad \text{and} \quad x = L, \quad \theta(L,t) = \theta_L = T_L - T_\infty$$

The initial condition (IC) is $\theta(x,0) = 0$ for $0 \le x \le L$.

To obtain the solution of the nonhomogeneous PDE with nonhomogeneous BC let the solution be written as

$$\theta(x,t) = v(x) + w(x,t)$$

where $v(x)$ represents the steady-state solution which will take care of the nonhomogeneous term in the PDE and the nonhomogeneous BC, and $w(x,t)$ is the solution of the associated homogeneous PDE:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha}\frac{\partial w}{\partial t}, \quad t > 0, \quad 0 < x < L$$

with homogeneous boundary conditions: $w_x(0,t) = 0$ and $w(L,t) = 0$.

(a) Find the ODE for $v(x)$, specify its BCs, then obtain its solution.

(b) Find the homogeneous PDE for $w(x, t)$, specify its homogenous BCs, then obtain its solution by means of the Separation of Variables Method (SVM).

(c) Obtain the symbolic relation for the instantaneous heat transfer rate $Q(t)$ through the right boundary $x = L$ using Fourier's Law of Conduction:

$$Q(t) = -kA\left(\frac{\partial \theta}{\partial x}\right)_{x=L}$$

(d) Use a Computer Alegbra System such as Maple or Mathcad to plot the steady-state solution $v(\xi)$ where $\xi = x/L$ for the following system parameter values:

$D = 5\,mm$
$L = 100\,mm$
$k = 80\,W/(m \cdot K)$
$\alpha = 1.2 \times 10^{-5}\,m^2/s$
$S = 2 \times 10^6\,W/m^3$
$h = 20\,W/(m^2 \cdot K)$
$T_L = 70°C$
$T_\infty = 20°C$

(e) Plot the transient solution $T(\xi, \tau)$ on the interval $0 \leq \xi \leq 1$ for three different dimensionless times: $\tau = \alpha t/L^2 = 0.01, 0.1, \text{and } 1$ for the system parameter values given above.

(f) Compute $Q(\tau)$ for $\tau = 0.01, 0.1,$ and 1.
