UNIVERSITY OF WATERLOO
Department of Mechanical Engineering
ME 303 Advanced Engineering Mathematics

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Given the linear, second order nonhomogeneous PDE:

$$\frac{1}{r}\frac{\partial}{dr}\left(r\frac{\partial T}{\partial r}\right) + \frac{S}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \qquad t > 0, \qquad 0 < r < a$$

with parameters: $S\ [W/m^3]$, and $k\ [W/m \cdot K]$. The units of the dependent variable is $T\ [K]$, The units for the other variables are: $r\ [m]\,, t\ [s]$, and $a\ [m]$.

Boundary conditions and initial condition are

$$\frac{\partial T(0,t)}{\partial r} = 0, \qquad T(a,t) = 0 \quad \text{and} \quad T(r,0) = 0, \quad 0 \le r \le a$$

There is no convenient temperature scale and time scale in the problem statement.

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1) What are the units of α?
The units of the three terms are identical. The units can be obtained from the first or second term. The units of the second term (source term) of the PDE are $[W/m^3] \,/\, [W/m \cdot K] = [K/m^2]$.
Since the units of the transient term (energy storage term) are identical we have

$$\frac{1}{\alpha}\frac{[K]}{[s]} = \frac{[K]}{[m^2]}$$

The units of the thermophysical parameter α are $[m^2/s]$.

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2) Obtain nondimensional form of PDE, BCs and IC. Use the following dimensionless parameters:

$$\phi(\rho,\tau) = \frac{T(r,t)}{T_r}, \qquad \tau = \frac{t}{t_r}, \qquad \rho = \frac{r}{a}$$

where T_r is an arbitrary reference temperature, t_r is an arbitrary reference time, and a is the radius of the solid circular cylinder.

Nondimensionalize the first term. Set $T = \phi T_r$ where T_r is a constant.

$$\frac{\partial T}{\partial r} = \frac{\partial}{\partial \rho}(\phi T_r)\frac{\partial \rho}{\partial r} = \frac{T_r}{a}\frac{\partial \phi}{\partial \rho}$$

Now,

$$r\frac{\partial T}{\partial r} = (a\rho) \times \frac{T_r}{a}\frac{\partial \phi}{\partial \rho} = T_r \rho \frac{\partial \phi}{\partial \rho}$$

Therefore,

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{\partial}{\partial \rho}\left(\rho T_r \frac{\partial \phi}{\partial \rho}\right)\frac{\partial \rho}{\partial r} = \frac{T_r}{a}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \phi}{\partial \rho}\right)$$

Finally,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{a\rho}\frac{T_r}{a}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \phi}{\partial \rho}\right) = \frac{T_r}{a^2}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \phi}{\partial \rho}\right)$$

Note which parameters determine the units at each step of the nondimensionalization process.

Nondimensionalize the transient term in a similar manner.

$$\frac{\partial T}{\partial t} = \frac{\partial(\phi T_r)}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{T_r}{t_r}\frac{\partial \phi}{\partial \tau}$$

The units are determine by the parameters: T_r/t_r.

The PDE can now be written as

$$\frac{T_r}{a^2}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \phi}{\partial \rho}\right) + \frac{S}{k} = \frac{1}{\alpha}\frac{T_r}{t_r}\frac{\partial \phi}{\partial \tau}$$

The terms are still dimensional. Observe that the units of the first term are determined by the parameters T_r/a^2, and the units of the third term are determined by the parameters $T_r/(t_r\,\alpha)$. Multiple through by a^2/T_r to get the nondimensional form of the PDE

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \phi}{\partial \rho}\right) + \frac{Sa^2}{kT_r} = \frac{a^2}{\alpha t_r}\frac{\partial \phi}{\partial \tau}, \quad \tau > 0, \quad 0 < \rho < 1$$

Note that by inspection of the second and third terms of the PDE, we find that the units of the group $S\,a^2/k$ are $[K]$, and the units of the group a^2/α are $[s]$.

Tberefore $S\,a^2/k$ represents a temperature scale, and a^2/α represents a time scale of the system.

The nondimensional form of the BCs are

$$\frac{\partial T(0,t)}{\partial r} = \frac{\partial(T_r\phi(0,\tau))}{\partial\rho}\frac{\partial\rho}{\partial r} = \frac{T_r}{a}\frac{\partial(\phi(0,\tau))}{\partial\rho} = 0 \quad \text{or} \quad \frac{\partial(\phi(0,\tau))}{\partial\rho} = 0$$

and

$$T(a,t) = T_r\phi(1,\tau) = 0 \qquad \text{or} \quad \phi(1,\tau) = 0$$

The nondimensional form of the IC is

$$T(r,0) = T_r\phi(\rho,0) = 0, \quad 0 \le r \le a \qquad \text{or} \quad \phi(\rho,0) = 0, \quad 0 \le \rho \le 1$$

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3) Set the reference temperature: $T_r = Sa^2/k$ and the reference time: $t_r = a^2/\alpha$ in the PDE to obtain the final dimensionless form without parameters:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + 1 = \frac{\partial\phi}{\partial\tau}, \quad \tau > 0, \quad 0 < \rho < 1$$

with homogeneous BCs and IC:

$$\frac{\partial(\phi(0,\tau))}{\partial\rho} = 0, \quad \phi(1,\tau) = 0 \quad \text{and} \qquad \phi(\rho,0) = 0, \quad 0 \le \rho \le 1$$

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4) Obtain the solution of the *steady-state* case where $\partial\phi/\partial\tau = 0$.
The PDE becomes the ODE with $\phi(\rho)$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + 1 = 0, \quad 0 < \rho < 1$$

and homogeneous BCs:

$$\frac{\partial(\phi(0,\tau))}{\partial\rho} = 0, \quad \phi(1,\tau) = 0$$

This ODE can be integrated twice in a straight forward manner. Here are the steps:

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) = -\rho$$

Integrate once to get

$$\rho\frac{\partial\phi}{\partial\rho} = -\frac{\rho^2}{2} + C_1$$

Dividing by ρ gives the derivative:

$$\frac{\partial \phi}{\partial \rho} = -\frac{\rho}{2} + \frac{C_1}{\rho}$$

The first BC at $\rho = 0$ can be used now to eliminate the second term which is unbounded at $\rho = 0$. Set $C_1 = 0$ to give

$$\frac{\partial \phi}{\partial \rho} = -\frac{\rho}{2}$$

Integrate a second time to get

$$\phi = -\frac{\rho^2}{4} + C_2$$

Use the BC at $\rho = 1$ to find C_2.

$$0 = -\frac{1}{4} + C_2 \quad \text{therefore} \quad C_2 = \frac{1}{4}$$

The dimensionless steady-state solution is

$$\phi = \frac{1}{4} - \frac{\rho^2}{4}, \quad 0 \leq \rho \leq 1$$

The solution of the ODE can be obtained by means of other less direct procedures such that those presented in an ODE course.

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5) Find $\phi(0)$ and $T(0)$ from the solution.

$$\phi(0) = \frac{1}{4} \quad \text{and} \quad T(0) = T_r\,\phi(0) = \frac{Sa^2}{4k}$$

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6) Given the Fourier Law of Conduction at the cylinder boundary:

$$Q = -k2\pi a \frac{\partial T(r)}{\partial r}, \quad r = a$$

per unit length of the cylinder.
(a) Obtain the nondimensional form and call it $Q^\star$.
(b) Determine Q from the nondimensional solution.

$$Q = -k2\pi a \left(\frac{\partial T(r)}{\partial r}\right)_{r=a} = -k2\pi a \left(\frac{T_r}{a}\frac{\partial \phi}{\partial \rho}\right)_{\rho=1} = -2\pi S a^2 \left(\frac{\partial \phi}{\partial \rho}\right)_{\rho=1}$$

Now we can define a dimensionless heat transfer rate at the cylinder boundary:

$$Q^\star = \frac{Q}{2\pi S a^2} = -\left(\frac{\partial \phi}{\partial \rho}\right)_{\rho=1}$$

The dimensionless heat transfer rate can be obtained from the dimensionless solution:

$$Q^\star = -\left(\frac{\partial \phi}{\partial \rho}\right)_{\rho=1} = -\left(-\frac{\rho}{2}\right)_{\rho=1} = \frac{1}{2}$$

The dimensional heat transfer rate at the boundary is obtained from the definition and the above result:

$$Q = 2\pi S a^2 Q^\star = \pi a^2 S$$

per unit length of the cylinder.
