STURM-LIOUVILLE PROBLEM (SLP)

The separation of variables method when applied to **second-order linear homogeneous PDEs** frequently leads to **second-order homogeneous ODEs** of the type:

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + [q(x) + \lambda r(x)]\, y(x) = 0, \qquad a \le x \le b$$

or in the equivalent form

$$[p(x)y']' + [q(x) + \lambda r(x)]\, y = 0, \qquad a \le x \le b$$

where p, q and r are given functions of the independent variable x in the interval $a \le x \le b$, λ is a parameter, and $y(x)$ is the dependent variable. This equation is known as the **Sturm-Liouville Differential Equation (SLDE)**. It is said to be **regular** in the interval $[a, b]$ if $p(x)$ and $r(x)$ are positive in the interval. The $r(x)$ is called the weight function, and it appears in the **orthogonality relation** to be discussed below.
The general second-order differential equation of the form:

$$a_2(x)y''(x) + a_1(x)y'(x) + [a_0(x) + \lambda]\, y(x) = 0, \qquad a \le x \le b$$

can be rewritten in the **self-adjoint form** by letting

$$p(x) = e^{\int \frac{a_1(x)}{a_2(x)} dx}, \qquad q(x) = \frac{a_0(x)}{a_2(x)} p(x), \qquad \text{and} \qquad r(x) = \frac{p(x)}{a_2(x)}$$

For a given value of λ two linearly independent solutions of a regular **SLDE** exist in $[a, b]$.

The **Boundary-Value Problem (BVP)** containing the **SLDE**, $a \le x \le b$, along with the separated **homogeneous end conditions**:

1. $y_m(a) = y_m(b) = 0; \quad y_n(a) = y_n(b) = 0 \quad \text{for both} \quad m \quad \text{and} \quad n$

or

2. $y'_m(a) = y'_m(b) = 0; \quad y'_n(a) = y'_n(b) = 0 \quad \text{for both} \quad m \quad \text{and} \quad n$

or a linear combination of the above two homogeneous conditions:

$$3a. \qquad a_1 y_m(a) + a_2 y_m'(a) = b_1 y_n(a) + b_2 y_n'(a) = 0$$

and

$$3b. \qquad a_1 y_m(b) + a_2 y_m'(b) = b_1 y_n(b) + b_2 y_n'(b) = 0$$

where the indices m and n denote different solutions, forms a **Sturm-Liouville Problem SLP**.

If the coefficients a_1, a_2 and b_1, b_2 are real constants such that $a_1^2 + a_2^2 \neq 0$ and $b_1^2 + b_2^2 \neq 0$, and the **SLDE** is regular, then the problem is a regular **SLP**.

The trivial solution $y_m(x) = 0$ and $y_n(x) = 0$ satisfies the **SLP** for any value of the parameter λ.

Nontrivial solutions are called **eigenfunctions** or **characteristic functions** of the **SLP**.

The corresponding values of λ_m or λ_n for which the nontrivial solutions exist are known as **eigenvalues** or **characteristic values**.

1. All the **eigenvalues** λ are real.

2. There is an infinite set of **eigenvalues**:

$$\lambda_1 < \lambda_2 < \lambda_3 < \ \ldots \ < \lambda_n < \lambda_{n+1} < \ \ldots \ \to \infty$$

3. Corresponding to each **eigenvalue**, λ_n, there is one **eigenfunction** (i.e., a nonzero solution) denoted $y_n(x)$ (which is unique to within a multiplicative constant). $y_n(x)$ has exactly $n-1$ zeros for $a < x < b$.

4. If $y_n(x)$ and $y_m(x)$ are two **different eigenfunctions** (corresponding to $\lambda_n \neq \lambda_m$), then they are defined to be **orthogonal** with respect to the **weight function** $r(x)$ on the interval $a \leq x \leq b$; i.e., they satisfy:

$$\int_a^b r(x)y_n(x)y_m(x)dx = 0 \qquad \text{if} \qquad \lambda_n \neq \lambda_m$$

the so-called **orthogonality property of eigenfunctions**.

If the **eigenfunctions**: y_n and y_m corresponding to the **eigenvalues**: λ_n and λ_m respectively, are solutions of the **SLDEs**:

$$[py_n']' + [q + \lambda_n r]\, y_n = 0$$

and

$$[py_m']' + [q + \lambda_m r]\, y_m = 0$$

Multiplying the *first* **SLDE** by y_m and the *second* **SLDE** by y_n and then subtracting the *first* from the *second* gives:

$$[py_m']'\, y_n - [py_n']'\, y_m + (\lambda_m - \lambda_n)\, r y_n y_m = 0$$

However,

$$\frac{d}{dx}([py_m']\, y_n - [py_n']\, y_m) = [py_m']'\, y_n - [py_n']'\, y_m$$

Using this relationship we have

$$\frac{d}{dx}\left(p\left[y_m' y_n - y_n' y_m\right]\right) = (\lambda_n - \lambda_m)\, r y_n y_m$$

Integrating the last equation with respect to x over $a \leq x \leq b$ we find:

$$\left[p\left(y_m' y_n - y_n' y_m\right)\right]_a^b = (\lambda_n - \lambda_m)\int_a^b r y_n y_m dx$$

We note that the **eigenfunctions** y_n and y_m satisfy the **homogeneous end**

conditions:

$$a_1 y_n(a) + a_2 y_n'(a) = 0$$
$$a_1 y_m(a) + a_2 y_m'(a) = 0$$

and

$$b_1 y_n(b) + b_2 y_n'(b) = 0$$
$$b_1 y_m(b) + b_2 y_m'(b) = 0$$

Excluding the trivial case $a_1 = a_2 = b_1 = b_2 = 0$, then for the nontrivial solutions we must have:

$$y_m'(a) y_n(a) - y_n'(a) y_m(a) = 0$$

and

$$y_m'(b) y_n(b) - y_n'(b) y_m(b) = 0$$

The last equations allow us to write:

$$(\lambda_n - \lambda_m) \int_a^b r y_n y_m dx = 0$$

Since $\lambda_n \neq \lambda_m$, then

$$\int_a^b r y_n y_m dx = 0 \qquad \text{if} \qquad n \neq m$$

which is the required **orthogonality property of eigenfunctions**.

Example 1 Solve the following Sturm-Liouville problem:

$$y'' + \lambda y = 0, \qquad 0 \le x \le \pi, \qquad y(0) = 0, \qquad y(\pi) = 0$$

Solution 1 In this problem we have: $p(x) = 1, q(x) = 0, r(x) = 1, a = 0, b = \pi, a_1 = 1, a_2 = 0, b_1 = 1$ and $b_2 = 0$. The eigenvalues are $\lambda = n^2, n = 1, 2, 3, \ldots$. The eigenfunctions are

$$y_1 = \sin x, \qquad y_2 = \sin 2x, \qquad y_3 = \sin 3x, \ldots$$

and, in general we have

$$y_n = \sin nx, \qquad n = 1, 2, 3, \ldots$$

where the arbitrary constants have been set equal to one, since eigenfunctions are unique only to within a multiplicative constant.

Example 2 Solve the following Sturm-Liouville problem:

$$y'' + \lambda y = 0, \qquad 0 \le x \le 1, \qquad y(0) + y'(0) = 0, \qquad y(1) = 0$$

Solution 2 In this problem we have: $p(x) = 1, q(x) = 0, r(x) = 1, a = 0, b = 1, a_1 = a_2 = b_1 = 1$ and $b_2 = 0$. The eigenvalues are $\lambda = n^2, n = 1, 2, 3, \ldots$. If $\lambda < 0$, the solution is trivial. If $\lambda = 0$, then $y = c_1 + c_2 x$, and the boundary conditions applied to this function show that an eigenfunction associated with the eigenvalue $\lambda = 0$ is $1 - x$.

If $\lambda > 0$, we have as the solution of the differential equation

$$y = c_1 \cos \sqrt{\lambda}\, x + c_2 \sin \sqrt{\lambda}\, x$$

The condition $y(0) + y'(0) = 0$ implies that $c_1 + c_2\sqrt{\lambda} = 0$, i.e., $c_1 = -c_2\sqrt{\lambda}$. The condition $y(1) = 0$ implies that $\sqrt{\lambda} = \tan \sqrt{\lambda}$. Thus the eigenvalues are the squares of the solutions of the transcendental equation $z = \tan z$ which must be solved numerically.