CLASSIFICATION OF LINEAR PDEs OF SECOND ORDER

Second order linear **PDEs** with two independent variables (x, y) have the general form:

$$\boxed{Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G}$$

where the coefficients A, B, C, D, E, F, and G are functions of x and y or they could be constants. The **PDEs** can be classified as either:

1. Hyperbolic if $B^2 - 4AC > 0$
2. Parabolic if $B^2 - 4AC = 0$
3. Elliptic if $B^2 - 4AC < 0$

The **PDEs** are defined to be *homogeneous* if $G = 0$, otherwise they are defined to be *nonhomogeneous*.

The **PDEs** are defined to be *nonlinear* if they contain terms like

$$u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial u^2}{\partial x}$$

EXAMPLES OF HYPERBOLIC, PARABOLIC, AND ELLIPTIC PDEs

1. The *Diffusion (Heat) Equation*

$$u_{xx} = u_t$$

is a second-order linear **PDE** with coefficients:

$$A = 1 \qquad B = 0 \qquad C = 0 \qquad D = 0$$

$$E = -1 \qquad F = 0 \qquad G = 0$$

Therefore

$$B^2 - 4AC = 0$$

for all x and t. The *Diffusion Equation* is *Parabolic.*

2. The *Wave Equation*

$$u_{xx} = u_{tt}$$

is a second-order linear **PDE** with coefficients:

$$A = 1 \qquad B = 0 \qquad C = -1 \qquad D = 0$$

$$E = 0 \qquad F = 0 \qquad G = 0$$

Therefore

$$B^2 - 4AC = 4 > 0$$

for all x and t. The *Wave Equation* is *Hyperbolic.*

3. The *Laplace Equation*

$$u_{xx} + u_{yy} = 0$$

is a second-order linear **PDE** with coefficients:

$$A = 1 \qquad B = 0 \qquad C = 1 \qquad D = 0$$

$$E = 0 \qquad F = 0 \qquad G = 0$$

Therefore

$$B^2 - 4AC = -4 < 0$$

for all x and y. The *Laplace Equation* is *Elliptic.*