ME 303 Advanced Engineering Mathematics

Fourier Cosine and Sine Series

M.M. Yovanovich

Many physical problems in engineering and science are solved by the use of **Fourier series**. The Fourier series of a periodic function $f(x)$ with period $2L$ is defined as the trigonometric series

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

where the Fourier coefficients A_0, A_n, B_n are given by

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \tag{2}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx \tag{3}$$

and

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \tag{4}$$

The kth term of the Fourier cosine and sine series are

$$A_k \cos\left(\frac{k\pi x}{L}\right) \quad \text{and} \quad B_k \sin\left(\frac{k\pi x}{L}\right) \tag{5}$$

The kth partial sum of the Fourier series is

$$f(x) \approx A_0 + \sum_{n=1}^{k} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{k} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{6}$$

Othogonality of Cosine and Sine Functions

The family of cosine functions $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ satisfies the orthogonality relation

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \neq 0 \\ L, & m = n = 0 \end{cases} \tag{7}$$

The family of sine functions $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ satisfies the orthogonality relation

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases} \tag{8}$$

The following properties of cosine and sine functions when $n = 0, 1, 2, 3 \ldots$ can be used to simplify the Fourier coefficients:

$$\sin(n\pi) = 0 \quad \text{and} \quad \sin(-n\pi) = 0 \tag{9}$$

and

$$\cos(n\pi) = (-1)^n \quad \text{and} \quad \cos(-n\pi) = 0 \quad n \neq 0 \tag{10}$$

Even and Odd Functions

The Fourier series expansion can be obtained with less effort if we take advantage of the even or odd property of the function. An even function requires that

$$f(-x) = f(x) \tag{11}$$

which means that the graph of the function is symmetric with respect to the vertical axis. For an odd function,

$$f(-x) = -f(x) \tag{12}$$

The cosine function is an even function, whereas the sine function is an odd function.

If $f(x)$ is an even function,

$$\int_L^L f(x)\,dx = 2\int_0^L f(x)\,dx \tag{13}$$

If $f(x)$ is an odd function,

$$\int_{-L}^L f(x)\,dx = 0 \tag{14}$$

The product of even and odd functions is and odd function, whereas the products of two even functions or two odd functions produce even functions.

An even function $f(x)$ has the Fourier expansion

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \qquad 0 < x < L \tag{15}$$

where

$$A_0 = \frac{2}{L}\int_0^L f(x)\,dx, \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \qquad n = 1, 2, 3, \cdots \tag{16}$$

For an odd function $f(x)$, the Fourier series has the form

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \qquad 0 < x < L \tag{17}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \qquad n = 1, 2, 3, \cdots \tag{18}$$

Fourier Coefficients of Several Functions

Sawtooth Wave

The first example is the periodic sawtooth wave defined as

$$f(x) = x, \qquad -L < x < L$$

with the periodic condition

$$f(x + 2L) = f(x)$$

This function, as defined, is an odd function.

The Fourier coefficients are found to be

$$A_n = 0 \qquad \text{for} \qquad n = 0, 1, 2, \ldots$$

and

$$B_n = -(-1)^n \left(\frac{2L}{n\pi}\right) \qquad \text{for} \qquad n = 1, 2, 3, \ldots$$

and therefore the sawtooth function can be represented by the Fourier series expansion:

$$f(x) = \frac{2L}{\pi}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) - \ldots\right]$$

Each term of the expansion (called harmonic) has a larger frequency than the previous term, and all frequencies are multiples of a fundamental frequency that has the same period as the function $f(x)$. The absolute value of the amplitude of each term is smaller than the previous term.