ME 303 Advanced Engineering Mathematics

Nondimensional Diffusion Equation
Boundary Conditions and Initial Condition

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DIMENSIONLESS PDE, BCs and IC

To illustrate how a Partial Differential Equation (**PDE**), and its Boundary Conditions (**BCs**) and the Initial Condition (**IC**) can be made nondimensional, let us begin with the following *dimensional* problem, i.e., the one-dimensional diffusion equation (heat equation).

PDE

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad 0 < x < L$$

BCs

$$t > 0, \quad x = 0, \quad T = T_0 \quad \text{and} \quad x = L, \quad T = T_i < T_0$$

IC

$$t = 0, \quad 0 \leq x \leq L, \quad T = T_i$$

The units of the dependent variable T is K, and the units of the independent variables x and t are m and s respectively. The units of the physical parameter $\alpha = k/(\rho c_p)$ are m^2/s. Therefore the units of each term of the **PDE** are K/m^2.

We begin the nondimensionalization process by considering the dimensionless position first which is defined as:

$$\xi = \frac{x}{L} \quad \text{where} \quad 0 \leq \xi \leq 1$$

Next we consider the dimensionless dependent variable (temperature):

$$\phi = \frac{T(x,t) - T_i}{T_0 - T_i} \quad \text{where} \quad 0 \leq \phi \leq 1$$

The dependent variable can be written as

$$T(x,t) = T_i + (T_0 - T_i)\phi$$

The first partial derivative of $T(x,t)$ with respect to the position variable can be written as

$$\frac{\partial T}{\partial x} = (T_0 - T_i)\frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{(T_0 - T_i)}{L}\frac{\partial \phi}{\partial \xi}$$

because $\partial\xi/\partial x = 1/L$.

The second partial derivative of $T(x,t)$ can be written as

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial T}{\partial x} = \frac{\partial}{\partial \xi}\left(\frac{(T_0 - T_i)}{L}\frac{\partial \phi}{\partial \xi}\right)\frac{\partial \xi}{\partial x} = \frac{(T_0 - T_i)}{L^2}\frac{\partial^2 \phi}{\partial \xi^2}$$

The first partial derivative of $T(x,t)$ with respect to time can be written as

$$\frac{\partial T}{\partial t} = (T_0 - T_i)\frac{\partial \phi}{\partial t}$$

The **PDE** can now be expressed as

$$\frac{\partial^2 \phi}{\partial \xi^2} = \frac{L^2}{\alpha}\frac{\partial \phi}{\partial t}$$

Since the left hand side of the above equation is dimensionless, the right hand side must also be dimensionless. Therefore the combinations:

$$\frac{L^2}{\alpha t} \quad \text{or} \quad \frac{\alpha t}{L^2}$$

must form dimensionless groups. The second group is called the dimensionless time; it will be denoted as $\tau = (\alpha t)/L^2$, and it will be used to nondimensionalize the **PDE**, **BCs** and the **IC**.

The **PDE**, **BCs** and the **IC** can now be expressed in the following dimensionless form:

PDE

$$\frac{\partial^2 \phi}{\partial \xi^2} = \frac{\partial \phi}{\partial \tau} \qquad 0 \leq \xi \leq 1$$

BCs

$$\tau > 0, \quad \xi = 0, \quad \phi = 1 \ \text{ and } \ \xi = 1, \quad \phi = 0$$

IC

$$\tau = 0, \quad 0 \leq \xi \leq 1, \quad \phi = 0$$

We observe that the initial condition is homogeneous and one of the two boundary conditions is nonhomogeneous. The steady-state solution is linear with respect to the space variable.

The major advantage to the nondimensional form of the **PDE**, **BCs** and the **IC** is the reduction in the number of independent variables. In the dimensional form the dependent variable $T = T(x, L, \alpha, t, T_i, T_0)$ is a function of six independent variables; and in the dimensionless form, the dependent variable $\phi = \phi(\xi, \tau)$ is a function of two independent variables: the dimensionless position ξ, and the dimensionless time τ. This is a significant simplification of the problem.

Another advantage to the dimensionless form is that it now applies to several other physical problems such as mass transfer and momentum transfer.