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## Week 11

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### Lecture 1

• Similarity Method Continued.
• Characteristics of $erf(\eta)$ and $erfc(\eta)$
• See ME 303 Web site for Maple worksheets on these special functions.
Further discussions of the solution for the 1D diffusion equation in half-space. The dimensional solution depends on several parameters: $T = f(x, t, \alpha, T_i, T_0)$ while the dimensionless solution depends on one parameter only: $\phi = f(\eta)$. The gradient at the free surface $x = 0$ or $\eta = 0$ is

$$\frac{d\phi}{d\eta} = \left[\frac{d}{d\eta} erfc(\eta)\right]_{\eta=0} = \left[-\frac{2}{\sqrt{\pi}} e^{-\eta^2}\right]_{\eta=0} = -\frac{2}{\sqrt{\pi}}$$

The solution $\phi$ approaches zero asymptotically. At $\eta = 2$, $\phi = 0.0109$ which is nearly zero for engineering applications. A penetration depth can be defined to be $\delta = 4\sqrt{\alpha t}$, the distance into the half-space where $\phi$ has dropped to the value 0.011.

The solution presented here also appears in mass transfer and momentum transfer. The transport parameter $\alpha$ called the thermal diffusivity in heat conduction is $\nu = \mu/\rho$, the kinematic viscosity in fluid mechanics and $D$, is called the mass diffusion coefficient in mass transfer.

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The following types of problems which can be handled.
• Given $(x, \alpha, t, T_i, T_0)$, find $T(x, t)$
• Given $(T_i, T_0, T(x, t), x, \alpha)$, find $t$
• Given $(T_i, T_0, T(x, t), x, t)$, find $\alpha$
• Given $(T_i, T_0, T(x, t), t, \alpha)$, find $x$

The last three problems require the inverse solution, i.e. given $\phi$ find $\eta$. This can be accomplished easily by means of Maple. See the Maple worksheet which deals with a mass transfer problem.

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Simple approximations for the error and complementary error functions for calculators.

Reference: P.R. Greene, *J. Fluids Engineering*, Vol. 111, 1989, pp. 224-226.

$$erf(x) = 1 - A\exp(-B(x+C)^2), \quad x \geq 0$$

and

$$erfc(x) = A\exp(-B(x+C)^2), \qquad x \geq 0$$

with correlation coefficients: $A = 1.5577, B = 0.7182, C = 0.7856$. The accuracy is reported to be within 0.42%.
If $y = erfc(x)$, the inverse complementary error function is $x = erfc^{-1}(y)$. where $0 < y < 1$ and $0 < x < \infty$.
The inverse is approximated by

$$x = -C + \sqrt{-\frac{\ln(\frac{y}{A})}{B}}$$

• See ME 303 Web site for note on polynomial approximations of error function and its inverse.

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**Lecture 2**

This material is not in the text. It will appear in the ME 353 course.
Solutions for 1D diffusion equation in half-space for Neumann and Robin boundary conditions. The initial condition and the boundary condition as $x \to \infty$ are the same for both solutions.

**Neumann solution**.
For $t > 0$ at $x = 0$, when $\partial\theta/\partial x = -q_0/k$, the solution is

$$\frac{k\,[T(x,t) - T_i]}{2q_0\sqrt{\alpha t}} = \frac{1}{\sqrt{\pi}}e^{-\eta^2} + \eta\; erfc(\eta)$$

where $\eta = x/(2\sqrt{\alpha t})$. The temperature at the surface changes with time, however, the temperature gradient at the surface is constant.
The surface temperature rise is given by

$$\frac{k\,[T(0,t) - T_i]}{2q_0\sqrt{\alpha t}} = \frac{1}{\sqrt{\pi}}$$

**Robin Solution**.
For $t > 0$ at $x = 0$, $\partial T/\partial x = -h/k\,[T_f - T(0,t)]$, the solution is

$$\frac{T(x,t) - T_i}{T_f - T_i} = erfc(\eta) - \exp\left(2\eta Bi + Bi^2\right)\; erfc\,(\eta + Bi)$$

where $Bi = (h/k)\sqrt{\alpha t}$ and $h$ is the heat transfer coefficient, $k$ is the thermal conductivity of the half-space, $T_f$ is the fluid temperature, $T_i$ is the initial temperature, and $\eta = x/(2\sqrt{\alpha t})$ as before. In this solution the surface temperature and the temperature gradient at the surface change with time.
The surface temperature rise is given by

$$\frac{T(0,t) - T_i}{T_f - T_i} = 1 - \exp\left(Bi^2\right) erfc\,(Bi)$$

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Transform Methods. There are several:
- Laplace Transform
- Fourier Sine and Cosine Transforms
- Hankel Transform (problems formulated in polar coordinates)
- Mellin Transform (problems formulated in spherical coordinates)
- Other less common transforms are available

- Laplace Transform Method will be considered in this course.

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**Laplace Transform Method**

- See the notes on the ME 303 Website and the Maple worksheets which outline the Laplace Transform Method with applications to ODES and PDES.
- See the Text by M.R. Spiegel, Applied Differential Equations, Chapter 6.
- See internet for many Websites which contain much information on ODEs, Laplace Transforms and Maple.
- See M.R. Spiegel, Mathematical Handbook, Shaum's Outline Series for definitions and Table of Laplace Transforms.

Chapter 6 of Spiegel Text.
- 1.1 Inroduction
- 1.2 Definitions and examples of Laplace Transforms; Transforms of derivatives; Application to ODEs; Dirac delta impulse function;
- 1.3 Properties of Laplace Transforms
- 1.4 Gamma functions and its properties.
- 1.6 Definitions of Heaviside unit step function.

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**Definitions of Laplace Transform and its Inverse Transform**.

The Laplace transform of $f(x,t)$ is defined as

$$\mathcal{L}\left\{f(x,t)\right\} = \int_0^\infty e^{-st}\, f(x,t)dt = F(x,s) \qquad s > 0$$

and its inverse is defined as

$$\mathcal{L}^{-1}\left\{F(x,s)\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(x,s)e^{st}\,ds = f(x,t)$$

• Because the inverse Laplace tranform is very difficult to get from the definition given above, we will use Tables and Computer Algebra Systems such as Maple to get the inverse.

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• Maple Package and Commands
• Read in the Maple worksheet the Laplace Transform Package:
− `with(inttrans)`
• Maple commands:
− `laplace(f(x,t),t,s)`
− `invlaplace(f(x,s),s,t)`

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• Some Laplace Transforms

$$\mathcal{L}\left\{a\right\} = \frac{a}{s}, \quad s > 0, \quad a = \text{constant}$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \quad s > a$$

$$\mathcal{L}\left\{\cos\omega t\right\} = \frac{s}{s^2+\omega^2}, \quad s > 0 \quad \text{and} \quad \mathcal{L}\left\{\sin\omega t\right\} = \frac{\omega}{s^2+\omega^2}, \quad s > 0$$

$$\mathcal{L}\left\{t\right\} = \frac{1}{s^2}, \quad s > 0$$

$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}, \quad s > 0, \quad n = 1, 2, 3, \ldots$$

See Tables of Laplace Transforms for other examples.

• Laplace Transform of Derivatives

$$\mathcal{L}\left\{Y'(t)\right\} = sy(s) - Y(0)$$

where $Y(0)$ is the initial value of the function, and

$$y(s) = \mathcal{L}\left\{Y(t)\right\}$$

$$\mathcal{L}\left\{Y''(t)\right\} = s^2y(s) - sY(0) - Y'(0)$$

where $Y(0)$ is the initial value of the function, and $Y'(0))$ is the initial value of its derivative. Higher order derivatives can also be transformed. These are important in the Laplace transform of ODEs.

• Properties of Laplace Transform Operator
− It is a linear operator

$$\mathcal{L}\left\{af(t)+bg(t)\right\}=a\,\mathcal{L}\left\{f(t)\right\}+b\,\mathcal{L}\left\{g(t)\right\}=a\,f(s)+b\,g(s)$$

• Example

$$\mathcal{L}\left\{3-5e^{2t}+4\sin t-7\cos 3t\right\}=\mathcal{L}\left\{3\right\}+\mathcal{L}\left\{-5e^{2t}\right\}+\mathcal{L}\left\{4\sin t\right\}+\mathcal{L}\left\{-7\cos 3t\right\}$$

and therefore

$$=3\mathcal{L}\left\{1\right\}-5\mathcal{L}\left\{e^{2t}\right\}+4\mathcal{L}\left\{\sin t\right\}-7\mathcal{L}\left\{\cos 3t\right\}$$

and finally we get

$$=\frac{3}{s}-\frac{5}{s-2}+\frac{4}{s^2+1}-\frac{7s}{s^2+9},\quad s>2$$

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**Lecture 3**

**Laplace Transform Method**

• Problems from Chapter 6 of the Spiegel Text
− Page 265: A exercises: 1,7
− Page 266: B exercises: 5
− Page 283: A excercises: 1,3,5
− 284: B excercises: 3,6
− 296: A excercises: 1,2,3

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• Gamma Function
• Definition

$$\Gamma(x+1)=\int_0^\infty \beta^x e^{-\beta}d\,\beta$$

where $\beta$ is a dummy variable. See Shaum's Outline Series: pages 101-102 for definition, plots and properties.

• Recurrence Formula

$$\Gamma(x+1)=x\Gamma(x)$$

• Values of Gamma function

$$\Gamma(1)=1$$

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1$$

$$\Gamma(3) = \Gamma(1+2) = 2\Gamma(2) = 2\Gamma(1+1) = 2 \cdot 1\Gamma(1) = 2!$$

and so on.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

In general we find

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2\,m-1)}{2^m}\sqrt{\pi}, \quad 1, 2, 3, \ldots$$

The Gamma function is singular at all negative integers: ($x = -1, -2, -3, \ldots$), and $x = 0$.

• For Negative Values of $x$.

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \quad x < 0$$

$$\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma(\frac{1}{2})}{s^{1/2}} = \sqrt{\frac{\pi}{s}}, \quad s > 0$$

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• Discuss the Laplace Transform of Heaviside unit step function and Dirac delta function. Application to ODEs. Definitions of Heaviside unit step function: $H(t-a)$ or $u(t-a)$ and Dirac delta function $\delta(t-a)$ also called the impulse function.

• Maple supports these two functions; they are called `Heaviside(t)` and `Dirac(t)`.

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**Properties of Heaviside and Dirac functions**.

$$\frac{d}{dt}H(t) = \delta(t)$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-a)dt = 1$$

**Laplace Transforms of Heaviside and Dirac functions**.

$$\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}, \quad s > 0$$

and

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}, \quad a > 0$$

and

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s), \quad a > 0$$

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Many discontinuous functions can be generated with the Heaviside function. Some examples are:

$$f(t) = H(t-a) - H(t-b)$$

and

$$f(t) = e^t \left[H(t-a) - H(t-b)\right]$$

and

$$f(t) = H(t-a) - H(a-t)$$

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- Laplace Transform Method: First Order ODE.
- Example of simple first order ODE from transient conduction.

$$\frac{d\theta}{dt} + m\,\theta = 0, \quad t > 0, \quad \text{IC} \quad \theta(0) = \theta_i$$

where $\theta(t) = T(t) - T_\infty$ and $m = hA/(\rho c_p V)$, and $A$ = surface area, and $V$ = volume of system. Units of $m$ are $[1/s]$.

- Solution by SVM

$$\frac{d\theta}{\theta} = -m\,dt, \quad \text{integration gives} \quad \ln\theta = -mt + \ln C_1$$

With IC we find

$$C_1 = \ln\theta_i, \quad \text{solution is} \quad \frac{\theta}{\theta_i} = e^{-mt}$$

- Laplace Transform Method

$$\mathcal{L}\left\{\frac{d\theta}{dt}\right\} - m\mathcal{L}\left\{\theta\right\} = 0$$

This gives

$$s\bar{\theta}(s) - \theta(0) - m\bar{\theta}(s) = 0 \quad \text{with IC} \quad s\bar{\theta}(s) - \theta_i - m\bar{\theta}(s) = 0$$

where the Laplace transform of the dependent variable is defined as

$$\mathcal{L}\left\{\theta(t)\right\} = \bar{\theta}(s)$$

Solving for $\bar{\theta}(s)$ we find the solution in the transform domain:

$$\bar{\theta}(s) = \frac{\theta_i}{s+m}$$

The solution is obtained by taking the inverse Laplace transform:

$$\theta(t) = \mathcal{L}^{-1}\left\{\bar{\theta}(s)\right\} = \theta_i \mathcal{L}^{-1}\left\{\frac{1}{s+m}\right\}$$

From Laplace Transform Tables we get the solution:

$$\theta(t) = \theta_i e^{-mt}$$