Solutions for Midterm Examination, February 7, 1995

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Problem 1.

Given PDE:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial y}, \qquad 0 < x < \pi, \qquad y > 0$$

with homogeneous boundary conditions:

$$w(0, y) = 0, \qquad w(\pi, y) = 0$$

and initial condition:

$$w(x, 0) = 150 \sin x - \frac{150}{3} \sin 3x, \qquad 0 \le x \le \pi$$

Use Separation of Variables Method (SVM). Let

$$w(x, y) = X(x) Y(y)$$

Substitution into PDE, followed by division by $X(x)Y(y)$ gives:

$$\frac{X''}{X} = \frac{Y''}{Y}$$

Let lhs = rhs = $0, -\lambda^2, \lambda^2$.
Option 1: 0:

$$X'' = 0, \Longrightarrow X(x) = Ax + B$$

and

$$Y' = 0, \Longrightarrow Y(y) = C$$

Application of homogeneous BCs requires:

$$X(0) = A \cdot 0 + B = 0, \qquad \text{and} \qquad X(\pi) = A \cdot \pi + B = 0$$

Therefore $A = B = 0$ which gives the trivial solution: $w(x, y) = 0$.

Option 2: $-\lambda^2$:

$$X'' + \lambda^2 X = 0, \Longrightarrow X(x) = A \cos \lambda x + B \sin \lambda x$$

and

$$Y' + \lambda^2 Y = 0, \Longrightarrow Y(y) = C \exp(-\lambda y)$$

Application of homogeneous BCs requires:

$$X(0) = A\cos(\lambda \cdot 0) + B\sin(\lambda \cdot 0) = 0, \qquad X(\pi) = A\cos(\lambda \cdot \pi) + B\sin(\lambda \cdot \pi) = 0$$

Therefore $A = 0$ which eliminates the cosine function from the solution, and $B\sin(\lambda\pi) = 0$. If we set $B = 0$, this gives the trivial solution $w(x, y) = 0$. Therefore we choose $\sin(\lambda\pi) = 0$, which leads to the eigenvalues:

$$\lambda_n = n, \qquad n = 1, 2, 3, \ldots$$

Superposition of all possible solutions gives:

$$w(x, y) = \sum_{n=1}^{\infty} B_n \exp(-n^2 y) \sin(nx)$$

Apply the initial condition to find the Fourier coefficients B_n.

$$w(x, 0) = \sum_{n=1}^{\infty} B_n \sin(nx) = 150 \sin x - \frac{150}{3} \sin 3x, \qquad 0 \leq x \leq \pi$$

This is a Fourier sine series on the finite interval $[0, \pi]$. Use the orthogonality property of sines to get for $m = n$:

$$B_n \int_0^{\pi} sin^2(nx) dx = \int_0^{\pi} 150 \sin(x) \sin(nx) dx + \int_0^{\pi} -\frac{150}{3} \sin(3x) \sin(nx) dx$$

Evaluation of the integrals gives:

$$\begin{array}{ll} n = 1, & B_1 = 150 \\ n = 2, & B_2 = 0 \\ n = 3, & B_1 = -\frac{150}{3} \\ n \geq 4, & B_n = 0 \end{array}$$

The solution is

$$w(x, y) = 150 \sin(x) \exp(-y) - 50 \sin(3x) \exp(-9y)$$

Option 3: λ^2:

$$X'' - \lambda^2 X = 0, \Longrightarrow X(x) = A\cosh \lambda x + B \sinh \lambda x$$

and

$$Y' - \lambda^2 Y = 0, \Longrightarrow Y(y) = C\exp(\lambda y)$$

Application of homogeneous BCs requires:

$$X(0) = A\cosh(\lambda \cdot 0) + B\sinh(\lambda \cdot 0) = 0, \qquad \text{and} \qquad X(\pi) = A\cosh(\lambda \cdot \pi) + B\sinh(\lambda \cdot \pi) = 0$$

The first boundary condition requires $A = 0$ which eliminates the hyperbolic cosine function. The second boundary condition requires $B = 0$.

Therefore $A = B = 0$ which gives the trivial solution $w(x, y) = 0$.

Problem 2.

Given the Sturm-Liouville Problem (SLP):

$$Y'' + \gamma^2 Y = 0, \qquad 0 < y < H$$

with homogeneous Neumann and Robin BCs:

$$Y'(0) = 0, \qquad kY'(H) + hY(H) = 0$$

The separation constant is γ^2 and the thermophysical parameters are positive quantities.
Identify the eigenfunctions and describe how to find the eigenvalues.
Estimate the first three eigenvalues given: $h = 5000 W/m^2 \cdot K, k = 15 W/m \cdot K$ and $H = 1.5m$. When $hH/k > 200$, the boundary condition is nearly equivalent to the limiting case: $hH/k \to \infty$.
Solution of ODE is

$$Y(y) = A\cos(\gamma y) + B\sin(\gamma y)$$

and its derivative is

$$Y'(y) = -\gamma A\sin(\gamma y) + \gamma B\cos(\gamma y)$$

Apply BC at $y = 0$:

$$Y'(0) = -\gamma A\sin(\gamma \cdot 0) + \gamma B\cos(\gamma \cdot 0) = \gamma B = 0$$

Therefore $\boxed{B = 0}$, and the sine function is eliminated. Apply the second boundary condition at $y = H$:

$$-k\gamma A\sin(\gamma H) + hA\cos(\gamma h) = 0$$

The option $A = 0$ gives a trivial solution, therefore it is rejected. Multiplying through by H leads to the characteristic equation:

$$-\gamma H\sin(\gamma H) + hH/k\cos(\gamma H) = 0$$

Let $\delta = \gamma H$ and $Bi = hH/k$, and rewrite equation as

$$-\delta\sin(\delta) + Bi\cos(\delta) = 0$$

This equation has infinitely many positive roots for any value of the parameter Bi. The Newton-Raphson iterative method can be used to find the roots. The parameter value is $Bi = (5000)(1.5)/15 = 500 > 200$. The roots are close to the values of the roots corresponding to

$$\cos(\delta) = 0$$

The eigenvalues are

$$\delta_n = \frac{n\pi}{2}, \qquad n = 1, 3, 5, \ldots$$

The values of γ are obtained from

$$\gamma_n = \frac{n\pi}{3}, \qquad n = 1, 3, 5, \ldots$$

Problem 3.

Given the linear, second-order, nonhomogeneous PDE:

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) - \rho g - k\frac{\partial y}{\partial t} + F\cos\omega t = \rho\frac{\partial^2 y}{\partial t^2}, \qquad t > 0, \qquad 0 < x < L$$

where $y(x,t)$ is the displacement from equilibrium and the units of the parameters: $T[N], \rho[kg/m], x[m], y[m]$
Note that the density is per unit length.
(a) What are the units of each term of the PDE?
The units can be determined from the first term where $[1/m][N][m/m] = [N/m]$.

(b) What are the units of the physical parameters: g, k, F, ω?
Units of the product ρg are $[kg/m]g[units] = [N/m]$ which gives: $g[m/s^2]$
Units of the product $k\partial y/\partial t$ are $k[units][m/s]$ which gives: $k[Ns/m^2]$.
Units of $F[N/m]$.
Units of ω are $\omega[radians/s]$.

(c) Obtain the equation for the steady-state case (equilibrium): $y(x) = f(x)$, when $\partial y/\partial t = 0$, $\partial^2 y/\partial t^2 = 0$, and $F = 0$.
The PDE reduces to the nonhomogeneous, second-order ODE:

$$\frac{d^2 y}{dx^2} = \frac{\rho g}{T}, \qquad 0 < x < L$$

(d) Find the solution for the fixed ends: $y(0) = 0, y(L) = 0$, and obtain the relationship between the maximum displacement $y_{\max}$ and the physical parameters: ρ, T, g, L.
Integrating twice gives:

$$y(x) = \frac{\rho g}{T}\frac{x^2}{2} + Ax + B$$

The homogeneous BCs require:

$$y(0) = \frac{\rho g}{T}\frac{0^2}{2} + A \cdot 0 + B = 0$$

and

$$y(L) = \frac{\rho g}{T}\frac{L^2}{2} + AL + B = 0$$

The constants of integration are:

$$B = 0, \qquad A = -\frac{\rho g L}{2T}$$

The solution is

$$y(x) = -\frac{\rho g}{2T}\left[xL - x^2\right]$$

The maximum displacement from the horizontal position occurs at the midpoint by symmetry arguments. It is found to be:

$$y_{\max} = -\frac{\rho g L^2}{8T}$$