UNIVERSITY OF WATERLOO
DEPARTMENT OF MECHANICAL ENGINEERING
ME 305 PARTIAL DIFFERENTIAL EQUATIONS

WINTER 1995 February 7, 1995
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Open book examination. All aids are allowed. All questions must be answered and they are of equal value. Show all steps and state clearly all assumptions made. Material which is not legible will not be considered. Good luck.

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1. Use the *separation of variables method* **SVM** to find *all independent functions* $X(x)$ and $Y(y)$, such that $w = X(x) \cdot Y(y)$ satisfies the linear, homogeneous, parabolic **PDE**:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial y} \qquad 0 \le x \le \pi \qquad y > 0$$

Obtain the solution of the above equation which satisfies the homogeneous boundary conditions **BCs**:

$$w(0, y) = 0 \qquad \text{and} \qquad w(\pi, y) = 0$$

and the initial condition **IC**:

$$w(x, 0) = 200 \sin x - \frac{200}{3} \sin 3x \qquad 0 \le x \le \pi$$

2. Determine the *eigenfunctions* and the *eigenvalues* for the following Sturm-Liouville problem **SLP**:

$$Y'' + \gamma^2 \, Y = 0, \qquad 0 \le y \le H$$
$$-k \, Y'(0) + h \, Y(0) = 0 \qquad Y'(H) = 0$$

The separation constant is γ^2.

Determine the *first eigenvalue* correct to *six decimal places* given $h = 50 \; W/m^2 \cdot K$, $k = 150 \; W/m \cdot K$ and $H = 6.6 \; m$.

3. The following linear, non-homogeneous, second-order, partial differential equation **PDE** was obtained by the application of Newton's second law: $\sum$ Forces = mass $\times$ acceleration to a flexible, elastic wire of length L, linear mass density ρ which is taut and fixed at both ends: $x = 0$ and $x = L$. The motion takes place in a vertical plane parallel to the $y-$axis.

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) - \rho g - k\frac{\partial y}{\partial t} + F\,\cos\omega t = \rho\frac{\partial^2 y}{\partial t^2} \qquad t > 0 \qquad 0 \le x \le L$$

The first term on the left-hand side of the **PDE** represents the net vertical force acting on an element of the wire and the tension T in the wire is assumed to be constant. The second term represents the body force due to gravitational acceleration g.

The third term represents the viscous damping force per unit length of wire due to air resistance which is proportional to the wire velocity $\partial y/\partial t$ and the parameter k is a positive constant.

The fourth term is the forcing function due to some external periodic force per unit length of the wire.

The units of some of the the geometric and physical parameters are: T $[N]$, ρ $[kg/m]$, y $[m]$, x $[m]$, and t $[s]$. The mass density ρ is constant.

(a) What are the units of each term of the above general equation?

(b) What are the units of the parameters: g, k, F, ω?

(c) Obtain the equation for the steady-state case: $y(x) = f(x)$, i.e. when $\frac{\partial y}{\partial t} = 0$, $\frac{\partial^2 y}{\partial t^2} = 0$ and there is no external force acting on the wire, i.e. when $F = 0$.

(d) Find the solution $y(x)$ and obtain the relation between y_{max} and the physical and geometric parameters: ρ, T, g, L.