UNIVERSITY OF WATERLOO
DEPARTMENT OF MECHANICAL ENGINEERING
ME 305 PARTIAL DIFFERENTIAL EQUATIONS

WINTER 1994 | February 8, 1994
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Open book examination. All aids are allowed. All questions must be answered and they are of equal value. Show all steps and state clearly all assumptions made. Material which is not legible will not be considered. Good luck.

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1. The linear, homogeneous, second-order three dimensional partial differential equations **PDEs**:

$$\nabla^2 u = \frac{1}{c^2} u_{tt} \qquad \nabla^2 u = \frac{1}{\alpha} u_t \qquad t > 0$$

where the first **PDE** is the wave equation (hyperbolic type) and the second is the diffusion equation (parabolic type). The constants c^2 and α which appear in the **PDEs** are positive physical parameters.

The two **PDEs** can be *separated* into the space-dependent **PDE**:

$$\nabla^2 U + \lambda^2 U = 0$$

which is called the Helmholtz equation by the substitution:

$$u(x, y, z, t) = e^{i\omega t}\, U(x, y, z)$$

where $i = \sqrt{-1}$ and ω is some positive parameter. The separation constant in the above separated **PDE** is λ.

Make the substitutions and obtain two expressions for λ^2: one for the wave equation and one for the diffusion equation.

2. The *separation of variables method* **SVM** is an effective method of finding solutions to certain linear homogeneous partial differential equations.
Use the **SVM** to find *all independent functions* $X(x)$ and $Y(y)$, such that $u = X \cdot Y$ satisfies the linear, homogeneous **PDE**:

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

Obtain the solution of the above equation which satisfies the homogeneous boundary condition **BC**:

$$u(x, 0) = 4e^{-x}$$

3. The following linear, non-homogeneous, second-order, partial differential equation **PDE** was obtained by the application of Newton's second law: $\sum$ Forces = mass$\times$acceleration to a flexible, elastic wire of length L, linear mass density ρ which is taut and fixed at both ends: $x = 0$ and $x = L$. The motion takes place in a vertical plane parallel to the $y-$ axis.

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) - \rho g - k\frac{\partial y}{\partial t} + F\,\cos\omega t = \rho\frac{\partial^2 y}{\partial t^2} \qquad t > 0 \qquad 0 \le x \le L$$

The first term in the **PDE** is the net vertical force acting on an element of the wire and the tension T in the wire is assumed to be constant. The second term is the body force due to gravitational acceleration g. The third term is the viscous damping force per unit length of wire due to air resistance which is proportional to the wire velocity $\partial u/\partial t$ and the parameter k is a positive constant. The fourth term is the forcing function due to some external periodic force per unit length of the wire.

The units of some of the the geometric and physical parameters are: T $[N]$, ρ $[kg/m]$, y $[m]$, x $[m]$, and t $[s]$. The mass density is constant.

(a) What are the units of each term of the above general equation?

(b) What are the units of the parameters: g, k, F, ω?

Obtain the equation for the steady-state case: $y(x) = f(x)$, i.e. when $\frac{\partial y}{\partial t} = 0$, $\frac{\partial^2 y}{\partial t^2} = 0$ and there is no external force acting on the wire, i.e. when $F = 0$.

Find the solution $y(x)$ and obtain the relation between y_{max} and the physical and geometric parameters: ρ, T, g, L.