Solutions for Spring 1999 ME 303 Midterm Examination

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Question 1:

(a) Show that $\phi = -C \ln \sqrt{x^2+y^2}$ is the solution of $u_{xx} + u_{yy} = 0$.

$$\phi_x = \frac{1}{u}\frac{du}{dx} = \frac{-C}{\sqrt{x^2+y^2}} \frac{d(\sqrt{x^2+y^2})}{dx} = \frac{-C}{\sqrt{x^2+y^2}} \frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x) = -Cx(x^2+y^2)^{-1}$$

Therefore

$$\phi_{xx} = -C(x^2+y^2)^{-1} - (Cx)\left[-(x^2+y^2)^{-2}(2x)\right]$$

and

$$\phi_{xx} = \frac{-C}{(x^2+y^2)} - \frac{2Cx^2}{(x^2+y^2)^2} = \frac{-C(x^2+y^2)+2Cx^2}{(x^2+y^2)}$$

Similarly

$$\phi_{yy} = \frac{-C(x^2+y^2)+2Cy^2}{(x^2+y^2)}$$

Adding the two results gives

$$\phi_{xx} + \phi_{yy} = \frac{-C(x^2+y^2)+2Cx^2 - C(x^2+y^2)+2Cy^2}{(x^2+y^2)^2}$$

and

$$\phi_{xx} + \phi_{yy} = \frac{-2C(x^2+y^2)+2C(x^2+y^2)}{(x^2+y^2)^2} = 0$$

(b) Find u_r and u_θ.

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{1}{r}\frac{\partial(-C\theta)}{\partial \theta} = \frac{C}{r}$$

and

$$u_\theta = \frac{\partial \psi}{\partial r} = \frac{\partial(-C\theta)}{\partial r} = 0$$

The velocity field produced by a line source is radial.

(c) Volumetric flow rate from the line source.

$$Q = 2\pi r u_r = 2\pi r \frac{C}{r} = 2\pi C$$

which depends on the line source strength C.

Question 2:

(a) Units of the transient term of PDE:

$$\rho c_p A_c \frac{\partial u}{\partial t} = \left[\frac{kg}{m^3}\right]\left[\frac{W \cdot s}{kg \cdot K}\right]\left[m^2\right]\left[\frac{K}{s}\right] = \left[\frac{W}{m}\right]$$

(b) (i)

$$\frac{\partial^2 u}{\partial x^2} - \frac{V}{\alpha}\frac{\partial u}{\partial x} - m^2 u = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$

with

$$m^2 = \frac{hP}{kA_c} \quad \text{and} \quad \alpha = \frac{k}{\rho c_p}$$

Comparison with the general expression for PDEs:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

we find: $A = 1, B = 0, C = 0, D = -\frac{V}{\alpha}, E = -\frac{1}{\alpha}, F = m^2, G = 0$ with $y = t$. Therefore

$$B^2 - 4AC = 0^2 - (1)(0) = 0$$

This is a linear, second order homogeneous PDE of the parabolic type.

(c) Find the units of α, m^2, h/k and h and k from the rewritten equation.
The units of all terms of PDE are identical to units of the first term. The units are $\left[\frac{K}{m^2}\right]$.
Therefore

$$\frac{1}{\alpha}\left[\frac{K}{s}\right] = \left[\frac{K}{m^2}\right]$$

and

$$\alpha\left[\frac{m^2}{s}\right]$$

Also

$$\bar{m}^2 u = \bar{m}^2\,[K] = \left[\frac{K}{m^2}\right]$$

Units of $\bar{m}^2 = hP/(kA_c)$ are $1/m^2$. The units of the group h/k can be found from

$$\frac{h}{k}\frac{P}{A_c} = \frac{h}{k}\frac{[m]}{[m^2]} = \left[\frac{1}{m^2}\right]$$

The units of h/k are $1/m$. The units of h can be obtained from the term hPu

$$h\,[m]\,[K] = \left[\frac{W}{m}\right]$$

give

$$h\left[\frac{W}{m^2\cdot K}\right]$$

The units of the parameter k can now be obtained:

$$\frac{h}{k}=\frac{[W/m^2\cdot K]}{k}=\left[\frac{1}{m}\right]$$

gives

$$k\left[\frac{W}{m\cdot K}\right]$$

(d) Solution of ODE with $V=0, h>0, \partial u/\partial t=0$. Here $u(x)$.

$$\frac{d^2u}{dx^2}-m^2u=0$$

Solution

$$u(x)=C_1\cosh mx+C_2\sinh mx \quad \text{or} \quad u(x)=C_1e^{mx}+C_2e^{-mx}$$

The solution can also be obtained by several methods. One method is to use the $D=\dfrac{d}{dx}$ operator method. The ODE can be written as

$$D^2u-m^2u=0 \quad \text{or as} \quad \left(D^2-m^2\right)u=0 \quad \text{where} \quad u\neq 0$$

Therefore $D^2-m^2=0$ and $D=\pm m$. The solution is

$$u=C_1e^{mx}+C_2e^{-mx}$$

(e) Solution of ODE with $V>0, h=0, \partial u/\partial t=0$. Here $u(x)$.

$$\frac{d^2u}{dx^2}-\frac{V}{\alpha}\frac{du}{dx}=0$$

Reduce order of ODE by setting $w=du/dx$ to give

$$\frac{dw}{dx}-\frac{V}{\alpha}w=0$$

Separate variables:

$$\frac{dw}{w}=\frac{V}{\alpha}dx$$

Integrate to get

$$\ln w=\frac{V}{\alpha}x+\ln C_1 \quad \text{or} \quad \ln\frac{w}{C_1}=\frac{V}{\alpha}x$$

Note that I wrote the constant as $\ln C_1$ for convenience. Now write ODE as

$$w = \frac{du}{dx} = C_1 \exp(\frac{V}{\alpha}x)$$

Integrate to get solution:

$$u(x) = C_2 \exp(\frac{V}{\alpha}x) + C_3$$

with $C_2 = C_1\alpha/V$.

Alternative solution method of ODE. Write ODE as

$$D^2u - \frac{V}{\alpha}Du = 0 = \left(D^2 - \frac{V}{\alpha}D\right)u = 0, \quad \text{where} \quad u \neq 0$$

Therefore

$$D\left(D - \frac{V}{\alpha}\right) = 0 \quad \text{and} \quad D = 0, \quad D = \frac{V}{\alpha}$$

The solution is

$$u = C_1 e^{0x} + C_2 e^{\frac{V}{\alpha}x} = C_1 + C_2 e^{\frac{V}{\alpha}x}$$

(f) Use SVM to get the separated ODEs. Assume $u(x,t) = X(x)T(t)$ and substitute into PDE to get

$$X''T - \frac{V}{\alpha}X'T - m^2XT = \frac{1}{\alpha}XT'$$

Divide through by XT to get

$$\frac{X''}{X} - \frac{V}{\alpha}\frac{X'}{X} - m^2 = \frac{1}{\alpha}\frac{T'}{T}$$

The left and right hand sides must be equal to the same constant for all values of x and t.

Define the separation constant to get the time ODE: $T' + \lambda^2\alpha T = 0$. Set the left hand side to the constant $-\lambda^2$ to get the spatial (space dependent) ODE:

$$X'' - \frac{V}{\alpha}X' + (\lambda^2 - m^2)X = 0$$

Do not obtain the solution of this ODE.

Question 3:

Sturm-Liouville Problem.
Given ODE:

$$Y'' + \gamma^2 Y = 0, \quad 0 < y < H$$

with homogeneous Dirichlet and Robin BCs:

$$Y(0) = 0 \quad \text{and} \quad -kY'(H) = hY(H)$$

Solution and its derivative are

$$Y(y) = C_1 \cos(\gamma y) + C_2 \sin(\gamma y)$$

and

$$Y'(y) = -C_1 \gamma \sin(\gamma y) + C_2 \gamma \cos(\gamma y)$$

Apply BCS:
BC at $y = 0$ requires:

$$Y(0) = 0 = C_1 \cos(\gamma 0) + C_2 \sin(\gamma 0) = C_1$$

Setting $C_1 = 0$ removes the cosine function from the solution. Now

$$Y(y) = C_2 \sin(\gamma y) \quad \text{and} \quad Y'(y) = C_2 \gamma \cos(\gamma y)$$

BC at $y = H$ requires:

$$k C_2 \gamma \cos(\gamma H) + h C_2 \sin(\gamma H) = 0$$

Since $C_2 = 0$ gives a trivial solution, we must set

$$-\gamma \cos(\gamma H) = \frac{h}{k} \sin(\gamma H) \quad \text{or} \quad -\gamma H \cos(\gamma H) = \frac{hH}{k} \sin(\gamma H)$$

after multiplication by H to make the relation nondimensional.

(a) When $\frac{h}{k} = \infty$ or $Bi = hH/k = \infty$, then $\sin(\gamma H) = 0$. Therefore the eigenvalues are

$$\gamma_n = \frac{n\pi}{H}, \quad n = 1, 2, 3, \ldots$$

and the eigenfunctions are $Y_n = C_n \sin(\gamma_n y)$.

(b) When $\frac{h}{k} = 0$ or $Bi = hH/k = 0$, then $\gamma \cos(\gamma H) = 0$. Since $\gamma = 0$ gives a trivial solution, it will be rejected. Therefore we set $\cos(\gamma H) = 0$. Therefore the eigenvalues are

$$\gamma_n = \frac{(2n-1)\pi}{2H}, \quad n = 1, 2, 3, \ldots$$

and the eigenfunctions are $Y_n = C_n \sin(\gamma_n y)$.

(c) From the results found in parts (a) and (b), the first eigenvalue $\delta_1 = \gamma_1 H$ lies in the interval $\pi/2 < \delta_1 < \pi$. A simple plot will support this statement.

Verification that $\delta_1 = 2.0002$ when $Bi = 0.916$. By substitution we get:

$$-\delta_1 \cos(\delta_1) = -2.0002 \times \cos(2.0002) = -2.0002 \times (-0.416329) = 0.832741$$

and

$$Bi \sin(\delta_1) = 0.916 \times \sin(2.0002) = 0.916 \times (0.909214) = 0.832840$$

The two sides of the characteristic equation are very close in value.
