UNIVERSITY OF WATERLOO
DEPARTMENT OF MECHANICAL ENGINEERING
ME 305 PARTIAL DIFFERENTIAL EQUATIONS

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Open book examination. All aids are allowed. All questions must be answered and they are of equal value. Show all steps and state clearly all assumptions made. Material which is not legible will not be considered.

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1. The boundary-value problem:

$$\textbf{ODE}: \qquad x\frac{d}{dx}\left[x\frac{d\phi}{dx}\right] + \lambda^2\phi = 0, \qquad 1 \le x \le 2$$

$$\textbf{BCs}: \qquad \frac{d\phi(1)}{dx} = 0, \qquad \phi(2) = 0$$

is a regular Sturm-Liouville Problem (**SLP**).

i) Are the boundary conditions homogeneous **Dirichlet, Neumann** or **Robin**?

ii) Compare the given **ODE** with the definition of an **SLP** and *identify* the parameters: $p(x), q(x), r(x), a$ and b.

iii) If $\phi_m(x)$ and $\phi_n(x)$ are *eigenfunctions* corresponding to the *eigenvalues*: λ_m and λ_n respectively, what is the *orthogonality* relation for these *eigenfunctions* on the interval $1 \le x \le 2$?

iv) Demonstrate by *substitution* that the solution to the above **ODE** is

$$\phi(x) = A\cos(\lambda \ln x) + B\sin(\lambda \ln x)$$

where A and B are arbitrary constants of integration.

2. The following partial differential equation (**PDE**) occurs in transient heat conduction problems in which there is motion of material with constant velocity in the positive $x-$direction, convection losses, and energy storage:

$$kA_c\frac{\partial^2 u}{\partial x^2} - \rho c_p V A_c\frac{\partial u}{\partial x} - hPu = \rho c_p A_c\frac{\partial u}{\partial t} \qquad 0 \le x \le L \qquad t > 0$$

where k is the thermal conductivity, A_c is the cross-sectional area of the system, ρ is the mass density, c_p is the specific heat, V is the velocity, h is the convection heat transfer coefficient, and P is the perimeter of the system. The units of the dependent variable $u(x,t)$ and the independent variables x, t are K, m, s respectively.

i) Divide through by the product kA_c, introduce the thermophysical parameter $\alpha = k/(\rho c_p)$, and rewrite the **PDE**.

ii) From the *rewritten* **PDE** determine the units of α, the groups: (V/α), (hP/kA_c), and then the parameter (h/k). What are the units of the physical parameters: h and k?

iii) The boundary conditions are the following: When $x = 0$, $u = u_0$ and when $x = L$, $u = u_L > u_0$. *Nondimensionalize* the **PDE**, the boundary conditions, and the interval. Begin the process of nondimensionalization with the interval and the boundary conditions where it is desirable to have the dimensionless dependent variable (say ϕ) *homogeneous* at $x = 0$ and unity at $x = L$.
Let the dimensionless position be denoted by ζ and the dimensionless time by τ.

3. Use the method of *separation of variables* to find *all independent functions* $X(x)$ and $Y(y)$, such that $w = X \cdot Y$ satisfies the linear, homogeneous, parabolic **PDE**:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial y} \qquad 0 \le x \le \pi \qquad y > 0$$

Obtain the solution of the above equation which satisfies the homogeneous boundary conditions:

$w(x,y) = 0$ when $x = 0$ and π

$w(x,0) = 100 \sin x - \dfrac{100}{3} \sin 3x \quad$ when $y = 0$ and $0 \le x \le \pi$