ME 305 Final Exam Solutions, April 10, 1995

Problem 1 Solution

• (a) Units of LHS and RHS of given PDE. ϕ is dimensionless temperature.

$$LHS: \quad \rho\left[\frac{kg}{m^3}\right] c_p \left[\frac{W\cdot s}{kgK}\right] u_e \left[\frac{m}{s}\right]\left[\frac{-}{m^2}\right] = \left[\frac{W}{m^3K}\right]$$

$$RHS: \quad k\left[\frac{W}{mK}\right]\left[\frac{-}{m^2}\right] = \left[\frac{W}{m^3K}\right]$$

• (b) Define the similarity parameter η from simple dimensional analysis of PDE. Divide k by ρc_p and define $\alpha = k/\rho c_p$ which has units: $[m^2/s]$. Units of LHS and RHS are both $\left[\frac{-}{m^2}\right]$. Comparing denominators of LHS ($\alpha x/u_e$) and RHS (y^2) means that $y^2 \sim \alpha x/u_e$ and therefore

$$y \sim \sqrt{\frac{\alpha x}{u_e}}$$

Similarity parameter can be defined as

$$\eta = \frac{y}{\sqrt{\frac{\alpha x}{u_e}}} \quad \text{or} \quad \frac{y}{2\sqrt{\frac{\alpha x}{u_e}}}$$

Select the second definition. Both selections will give the same form of the solution, but the form of the ODE will be different.

• (c) Transformed ODE.

$$\frac{d^2\phi}{d\eta^2} + 2\eta\frac{d\phi}{d\eta} = 0, \quad \eta > 0$$

• (d) Using lecture notes, solution of ODE with given BCs: $\phi(0) = 1, \phi(\infty) = 0$ is

$$\phi(\eta) = erfc(\eta) = erfc\left(\frac{y}{2\sqrt{\alpha x/u_e}}\right)$$

Problem 2 Solution

• (a) Units are: $\nabla^2\,[m^{-2}]$; $u\,$[neutron density]; units of each term are the same as units of $\nabla^2 u\,$[neutron density$/m^2$]; units of B^2 are identical to units of ∇^2, therefore units of $B\,[m^{-1}]$. Units of $k\,[m^2/s]$ to make units of LHS consistent with RHS.

• (b) Substitution, differentiation and collection of terms yields the two PDEs given.

• (c) Solution by SVM of $\frac{1}{k}v_t = v_{xx},\ 0 < x < L,\ t > 0$ with BCs: $v(0,t) = v(L,t) = 0$ and IC: $v(x,0) = f(x),\ 0 \le x \le L$ is

$$v(x,t) = \sum_{n=1}^{\infty} E_n \exp(-n^2\pi^2 kt/L^2)\sin(n\pi x/L), \quad t > 0, \quad 0 < x < L$$

with Fourier coefficients obtained from IC:

$$E_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$$

Solution for neutron density is

$$u(x,t) = e^{B^2kt}v(x,t), \quad t > 0, \quad 0 < x < L$$

Problem 3 Solution

• $PDE : u_{xx} + u_{yy} = 0$
• SVM: Let $u(x,y) = X(x)Y(y)$
• Get $X''/X + Y''/Y = 0$
• Three options:
(i) $X''/X = 0,\ Y''/Y = 0$
(ii) $X''/X = -\lambda^2,\ Y''/Y = +\lambda^2$
(iii) $X''/X = +\lambda^2,\ Y''/Y = -\lambda^2$
• Homogeneous BCs along $x = 0$ and $x = a$ require option (ii)
• $X-$ode is $X'' + \lambda^2 X = 0$; $X-$sol is $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$
• $Y-$ode is $Y'' - \lambda^2 Y = 0$; $Y-$sol is $Y(y) = C\cosh(\lambda y) + D\sinh(\lambda y)$
• $u_x(0,y) = X'(0)Y(y) = 0$ requires $X'(0) = 0$ and $B = 0$
• $u_x(a,y) + h/ku(a,y) = X'(a)Y(y) + h/kX(a)Y(y) = 0$ requires
$X'(a) + h/kX(a) = 0$ which leads to $-\lambda\sin(\lambda a) + h/k\cos(\lambda a) = 0$ which gives the characteristic equation: $\lambda_n a\sin(\lambda_n a) = Bi\cos(\lambda_n a)$ with parameter $Bi = ha/k$ and $n = 1, 2, 3\ldots$
• $u_y(x,b) = X(x)Y'(b) = 0$ requires $Y'(b) = 0$ which gives
$C\lambda\sinh(\lambda b) + D\lambda\cosh(\lambda b) = 0$
• $C \ne 0, D \ne 0, \lambda \ne 0$ therefore we can solve for C or D
• Select $D = -C\tanh(\lambda b)$
• Let $E = AC, F = AD$
• Fundamental solution: $u_n(x,y) = E_n\cos(\lambda_n x)\left[\cosh(\lambda_n y) - \tanh(\lambda_n b)\sinh(\lambda_n b)\right]$
• Superposition gives solution:

$$u(x,y) = \sum_{n=1}^{\infty} E_n\cos(\lambda_n x)\left[\cosh(\lambda_n y) - \tanh(\lambda_n y)\sinh(\lambda_n b)\right]$$

and therefore,

$$u_y(x,y) = \sum_{n=1}^{\infty} E_n\cos(\lambda_n x)\left[\lambda_n\sinh(\lambda_n y) - \tanh(\lambda_n b)\lambda_n\cosh(\lambda_n y)\right]$$

- $u_y(x,0) = \sum_{n=1}^{\infty} -E_n \cos(\lambda_n x)\tanh(\lambda_n b)\lambda_n$
- Apply BC along $y = 0$ to find Fourier coefficients E_n.
- $\sum_{n=1}^{\infty} -E_n \cos(\lambda_n x)\tanh(\lambda_n b)\lambda_n = 0$ for $0 \le x < c$ and $= -q_0/k$ for $c < x \le a$
- Use orthogonality property of cosines to find E_n

$$E_n = \frac{\int_c^a (q_0/k)\cos(\lambda_n x)dx}{\lambda_n \tanh(\lambda_n b)\int_0^a \cos^2(\lambda_n x)dx}$$

- Evaluation of integrals gives:

$$E_n = \frac{2aq_0}{k}\frac{\sin(\delta_n) - \sin(c/a\delta_n)}{\delta_n \tanh(b/a\delta_n)\left[\delta_n + \cos(\delta_n)\sin(\delta_n)\right]}$$

with $\delta_n = \lambda_n a$, and δ_n are roots of characteristic equation for given value of Bi.

Problem 4 Solution

- (a) Substitute $u(r,t) = v(r) + w(r,t)$ into PDE to get:

$v_{rr} + w_{rr} + v_r/r + w_r/r = (1/\nu)w_t + (1/\mu)\partial P/\partial z$
- Separate into

nonhomogeneous ODE: $v_{rr} + (1/r)v_r = (1/\mu)\partial P/\partial z$ and

homogemeous PDE: $w_{rr} + (1/r)w_r = (1/\nu)w_t$
- (b) Separate PDE by $w(r,t) = R(r)\tau(t)$ into three sets of ODEs
- (i) $R'' + R'/r = 0$, $\tau' = 0$
- (ii) $R'' + R'/r + \lambda^2 R = 0$, $\tau' + \lambda^2 \nu\tau = 0$
- (iii) $R'' + R'/r - \lambda^2 R = 0$, $\tau' - \lambda^2 \nu\tau = 0$
- BCs are $u(0,t) = R(0)\tau(t) \ne \infty$ requires $R(0) \ne \infty$ and $u(a,t) = R(a)\tau(t) = 0$ requires $R(a) = 0$
- Solutions of $R-$odes are:
- (i) $R(r) = A + B\ln r$
- (ii) $R(r) = AJ_0(\lambda r) + BY_0(\lambda r)$
- (iii) $R(r) = AI_0(\lambda r) + BK_0(\lambda r)$
- $\ln r, Y_0(\lambda r), K_0(\lambda r)$ are singular at $r = 0$. They cannot appear in solution. Therefore $B = 0$ for all three options.
- Fundamental solution is $w_n(r,t) = E_n \exp(-\lambda_n^2 \alpha t)J_0(\lambda_n r)$
- Superposition gives: $w(r,t) = \sum_{n=1}^{\infty} E_n \exp(-\lambda_n^2 \alpha t)J_0(\lambda_n r)$
- λ_n are found from the zeros of $J_0(\lambda_n a) = 0$
- Solution of $v_{rr} + (1/r)v_r = (1/\mu)\partial P/\partial z$ with $v(0) \ne \infty$ and $v(a) = 0$ is obtained from $v(r) = -Sr^2/4 + C_1 + C_2 \ln r$ where $S = (1/\mu)\partial P/\partial z$ for convenience.
- Boundedness condition requires $C_2 = 0$ and $v(a) = 0$ requires $C_1 = -Sa^2/4$
- $v(r) = S(a^2 - r^2)/4$
- Initial condition $u(r,0) = v(r) + w(r,0) = 0$ requires that $w(r,0) = -v(r)$
- $w(r,0) = \sum_{n=1}^{\infty} E_n J_0(\lambda_n r) = -v(r), \quad 0 < r < a$
- Fourier-Bessel series leads to the Fourier-Bessel coefficients E_n.

Use orthogonality property of Bessel functions.

Multiply all terms under the summation and the RHS by $rJ_0(\lambda_m r)dr$ and integrate with respect to r from 0 to a to get

$$E_n = \frac{\int_0^a -v(r) r J_0(\lambda_n r) dr}{\int_0^a r J_0^2(\lambda_n r) dr}$$

The denominator integrates to:

$$\frac{a^2}{2}\left[J_0^2(\lambda_n a) + J_1^2(\lambda_n a)\right] = \frac{a^2}{2} J_1^2(\lambda_n a)$$

because $J_0(\lambda_n a) = 0$ which arises from homogeneous BC: $u(a,t) = 0$.
The numerator will not be integrated. This requires much effort and good tables of integrals of Bessel functions or a Computer Algebra System.

Problem 5 Solution

• PDE: $u_{yy} = (1/\nu) u_t, \quad t > 0, \quad 0 < y < L$
• IC: $u(y,0) = 0$, BCs: $u(0,t) = 0,\ u(L,t) = u_0$
• (a) Nondimensional velocity: $\phi = u/u_0$, nondimensional position: $\zeta = y/L$, nondimensional time: $\tau = \nu t/L^2$.
PDE: $\phi_{\zeta\zeta} = \phi_\tau,\ \tau > 0,\ 0 < \zeta < 1$.
IC: $\phi(\zeta, 0) = 0$; BCs: $\phi(0,\tau) = 0,\ \phi(1,\tau) = 1$
• (b) Laplace transform of dimensional form of PDE and BCs:

$$\text{PDE}: \qquad \frac{d^2\bar{u}}{dy^2} - \frac{s}{\nu}\bar{u} = 0$$

$$\text{BCs}: \qquad \bar{u}(0,s) = 0, \qquad \bar{u}(L,s) = \frac{u_0}{s}$$

• Solution: for finite region use hyperbolic form:
$\bar{u}(x,s) = A\cosh(\sqrt{s/\nu}\, x) + B\sinh(\sqrt{s/\nu}\, x)$
• Apply BCs to get:

$$\bar{u}(x,s) = u_0 \frac{\sinh(\sqrt{s/\nu} y)}{s\sinh(\sqrt{s/\nu} L)}$$

• (c) Apply Laplace transform to $\tau = \mu \partial u(L,t)/\partial y$ to get:

$$\bar{\tau} = \mu \frac{d\bar{u}(L,s)}{dy} = \frac{\mu u_0}{\sqrt{\nu}} \frac{1}{\sqrt{s}} \frac{1}{\tanh(\sqrt{s/\nu} L)}$$

• (d) From Laplace transform tables: Spiegel, Mathematical Handbook, item 128 with $x = y$ and $a = L$ the solution is

$$u(y,t) = u_0\left[\frac{y}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp(-n^2\pi^2\nu t/L^2)\sin(n\pi y/L)\right]$$