303FES98.TEX

UNIVERSITY OF WATERLOO

Department of Mechanical Engineering
ME 303 Advanced Engineering Mathematics

Spring Term 1998 — August 12, 1998
M.M. Yovanovich — 9:00–12:00 P.M.

Open book examination. Aids are allowed such as course text, lecture material and any material from the Website, calculator and Spiegel's Mathematical Handbook. All questions must be answered and they are of equal value.

Read each question carefully before beginning the analysis. Answer the questions which are asked.

Show all steps and state clearly all assumptions made. Material which is not legible will not be considered.

The problems deal with Laplace, Diffusion and Wave Equations which are classified as elliptic, parabolic and hyperbolic types. The solution methods employed are of the type: separation of variables, Laplace transform, similarity transformation or numerical.

The mathematical problems can come from several engineering areas such as: conduction heat transfer, mass transfer, fluid mechanics, and dynamics of solids.

Good luck.

Problem 1. (20)

(a) Apply the Separation of Variables Method (SVM) to obtain the solution $u(x, y)$ of the two-dimensional Laplace equation (elliptic type) in the semi-infinite region: $x > 0, 0 \leq y \leq b$:

$$PDE: \qquad u_{xx} + u_{yy} = 0$$

Homogeneous Dirichlet and Neumann boundary conditions are specified along the boundaries $y = 0$ and $y = b$, respectively:

$$u(x, 0) = 0, \qquad u_y(x, b) = 0$$

The solution must satisfy the regular condition at infinity:

$$u(x \to \infty), y) \to 0$$

The fourth nonhomogeneous Dirichlet boundary condition along $x = 0$ is

$$u(0, y) = u_0 \sin\left(\frac{\pi y}{2b}\right), \quad 0 \leq y \leq b$$

where u_0 is a positive constant.

(b) Use the solution to find:

$$Q_1 = \int_0^b -k u_x(0, y)\, dy, \quad \text{and} \quad Q_2 = \int_0^\infty k u_y(x, 0)\, dx$$

where k is the thermal conductivity of the region. You should find that $Q_1 = Q_2$.

Problem 2. (20)

The displacement $y(x, t)$ of an elastic string of length L, mass density per unit length ρ and constant tension T is described by the one-dimensional wave equation (hyperbolic type):

$$PDE: \qquad c^2 y_{xx} = y_{tt} + k y_t, \quad t > 0, \quad 0 < x < L$$

with the constants $c^2 = T/\rho > 0$ and $k > 0$. The viscous damping term $k y_t$ is proportional to the product of the local string velocity y_t and the coefficient k.

The homogeneous Dirichlet boundary conditions corresponding to fixed ends are:

$$y(0, t) = 0, \qquad y(L, t) = 0$$

The general initial conditions for all $t > 0$ are:

$$y(x, 0) = f(x), \qquad \text{and} \qquad y_t(x, 0) = g(x)$$

where $f(x)$ and $g(x)$ represent arbitrary initial displacement and velocity.

(a) Use the Separation of Variables Method with $u(x,t) = X(x)\tau(t)$ where $X(x)$ and $\tau(t)$ represent independent space and time functions to separate the PDE into two independent, but related, second-order ordinary differential equations. Before application of the SVM, divide through by c^2.

(b) Show that one of the separated equations with the homogeneous BCs is the regular Sturm-Liouville Problem (SLP):

$$X'' + \lambda^2 X = 0, \quad 0 < x < L$$

with BCs:

$$X(0) = 0, \qquad X(L) = 0$$

What is the solution of this SLP? What are the corresponding eigenfunctions and eigenvalues?

(c) What is the corresponding time dependent ODE? Obtain its general solution when $k = 0$.

(d) Obtain the solution of the wave equation for the case where: $f(x) = y_0 \sin(\pi x/L)$ and $g(x) = 0$ where $y_0/L << 1$.

(e) Write the non-dimensional form of the PDE, BCs and IC for the case where $k = 0$ and the conditions of part (d). Let the dimensionless displacement be ϕ, dimensionless position be ζ and dimensionless time be $\tau^\star$.

Problem 3. (20)

One-dimensional momentum diffusion is described by the following parabolic type, partial differential equation:

$$u_{yy} = \frac{1}{\nu}\, u_t \quad y > 0, \quad t > 0$$

where u is the velocity $[m/s]$, ν is the fluid kinematic viscosity $[m^2/s]$, y is the position $[m]$ and t is the time $[s]$.

The initial and boundary conditions are respectively,

$$u(y,0) = u_0, \qquad u(0,t) = 0, \qquad u(\infty,t) = u_0$$

(a) By means of the similarity parameter, $\eta = y/\sqrt{4\nu t}$, transform the partial differential equation into a second-order ordinary differential equation in which $u = u(\eta)$ and obtain its solution. Your solution must show all steps including the transformed initial and boundary conditions.

The instantaneous momentum flux is given by Newton's law of viscosity:

$$\tau(y,t) = \mu\, \frac{\partial u(y,t)}{\partial y}$$

where τ is the shear stress $[N/m^2]$, μ is the fluid viscosity $[kg/(m \cdot s)]$, and $\nu = \mu/\rho$ where ρ is the fluid mass density $[kg/m^3]$.

(b) Obtain the expression for the time dependent momentum flux at $y = 0$. ie, $\tau(0, t)$.

Problem 4. (20)

Consider the one-dimensional heat equation (parabolic type) in a bar of constant cross-sectional area A and length L:

$$\text{PDE}: \qquad u_{xx} = \frac{1}{\alpha} u_t, \quad t > 0, \quad 0 < x < L$$

where $\alpha > 0$ is called the thermal diffusivity in transient conduction problems. The diffusivity is constant.
The nonhomogeneous Dirichlet boundary conditions are:

$$\text{BCs}: \qquad u(0, t) = u_0, \quad \text{and} \quad u(L, t) = u_0$$

and the initial condition is

$$\text{IC}: \qquad u(x, 0) = u_0 + u_1 \sin\left(\pi \frac{x}{L}\right), \quad u_1 > u_0$$

The Laplace Transform Method (LTM) is well suited to handle problems which have non-homogenous initial and boundary conditions.

(a) Find the Laplace Tranform of the PDE and the BCs. Let $U(x, s)$ represent the Laplace transform of the solution $u(x, t)$.

(b) Obtain the solution $U(x, s)$ of the transformed nonhomogeneous second-order ODE.

Hint: The solution $y(x)$ of the nonhomogenous ODE:

$$y'' - my + a + b \sin\left(\frac{\pi x}{L}\right) = 0$$

is

$$y(x) = \frac{a}{m} + \left[\frac{b}{m + \pi^2/L^2}\right] \sin\left(\frac{\pi\, x}{L}\right) + C_1\, e^{\sqrt{m}x} + C_2\, e^{-\sqrt{m}x}$$

(c) Obtain the solution $u(x, t)$ by finding the inverse Laplace Transform of $U(x, s)$. Use Tables of Laplace Transforms.

Problem 5. (20)

Consider a circular rod with constant cross-section A, length L and thermal conductivity k. The end at $x = 0$ is maintained at constant temperature T_0, while the end at $x = L$ is adiabatic, therefore $T_x(L) = 0$. The sides of the rod are in contact with a fluid at temperature $T_f < T_0$, and there is heat transfer by convection from the rod sides into the surrounding fluid through a constant, uniform heat transfer coefficient h. This problem appears in the design of pin-fins for microelectronics cooling.

Introducing the temperature excess: $\theta(x) = T(x) - T_f$, the governing equation is the second-order ODE:

$$\theta_{xx} - m^2\theta = 0, \qquad 0 < x < L$$

with system parameter: $m^2 = hP/(kA) > 0$ where P is the perimeter of the rod.

The boundary conditions are:

$$\theta(0) = \theta_0, \quad \text{and} \quad \theta_x(L) = 0$$

The solution of the ODE is easily found to be

$$\theta(x) = \theta_0 \left[\frac{\cosh m(L-x)}{\cosh mL}\right], \qquad 0 < x < L$$

The heat transfer rate through the rod is obtained from application of Fourier's Law of Conduction at the end $x = 0$:

$$Q_{theo} = -kA\theta_x(0) = kAm\theta_0 \tanh mL = \sqrt{hPkA}\,\theta_0 \tanh mL$$

Apply the Control Volume Method to obtain a numerical solution of the ODE. Use three equal length control volumes, ie, $N = 3$. For each control volume (CV) use the approximation of Fourier's Law of Conduction: $Q_{12} = kA(\theta_1 - \theta_2)/(L/N)$ for example, for conduction from CV_1 into CV_2 and the approximation of Newton's Law of Cooling: $Q_1 = hP\theta_1 L/N$ for convective heat transfer from CV_1, for example. Provide a simple sketch of the system with three CVs, showing the heat conduction into and out of each CV, and the convective heat transfer rates from each CV.

(a) Let the temperature excess nodes be denoted as: $\theta_0, \theta_1, \theta_2, \theta_3, \theta_4$ where θ_0 and θ_4 are the end temperature nodes respectively, and θ_1 through θ_3 are the internal temperature nodes. Obtain equations for the temperature nodes with parameter: $m^2 = hP/(kA)$.

(b) Given the pin-fin parameters:
$L = 20\,mm$,
$D = 2\,mm$,
$k = 100\,W/m \cdot K$,
$h = 40\,W/m^2 \cdot K$,
$\theta_0 = 100\ ^\circ C$
obtain *three* equations for the temperature nodes: $\theta_1, \theta_2, \theta_3$.

(c) Solve the set of three equations to obtain numerical values for the temperature nodes: $\theta_1, \theta_2, \theta_3$. For your convenience the numerical value of the parameter: m^2L^2/N^2 which appears in each equation is reported to be 0.0355556 when $N = 3$.

(d) The convective heat transfer rate from the system is given approximately by the sum of the CV heat transfer rates:

$$Q_{conv} = Q_1 + Q_2 + Q_3 = \frac{hPL}{N}(\theta_1 + \theta_2 + \theta_3)$$

¿From the numerical results found in part (c), compute the heat transfer rate through the system.

The theoretical temperature excess at the three internal nodes are:
$\theta_1 = 95.6086$
$\theta_2 = 89.3465$
$\theta_3 = 86.2700$
and the theoretical system heat transfer rate is $Q_{theo} = 0.4551\,W$
