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General Solution of Poisson's Equation for Plane Wall, Long Solid Circular Cylinder and Solid Sphere

The general Poisson equation $\nabla^2 T = -S/k$ with appropriate boundary conditions, and the general solution which is valid for the plane wall, long solid circular cylinder and solid sphere is presented below. The temperature is one-dimensional, ie $T = T(u)$ where u represents the $x-$ and $r-$ coordinates in the plane wall, cylinder and sphere. The distributed heat source S and the thermal conductivity k are constant.

Differential Equation

$$\frac{1}{u^n}\frac{d}{du}\left(u^n \frac{dT}{du}\right) = -\frac{S}{k}, \qquad 0 \le u \le b, \qquad n = 0, 1, 2$$

Boundary Conditions

$$u = 0, \quad \frac{dT}{du} = 0, \qquad \text{symmetry condition}$$

$$u = b, \quad \frac{dT}{du} = -\frac{h}{k}\left[T(b) - T_f\right]$$

The parameters in the previous equations are defined below for the three geometries.

Geometry	Plane Wall	Cylinder	Sphere
$u =$	x	r	r
$b =$	L	b	b
$n =$	0	1	2

Solution

Integrating once gives the temperature gradient:

$$\frac{dT}{du} = -\frac{S}{k}\frac{u}{n+1} + \frac{C_1}{u^n}$$

The second integration gives the temperature distribution:

$$T(u) = -\frac{S}{k}\frac{u^2}{2(n+1)} + C_1\frac{u^{1-n}}{(1-n)} + C_2$$

To satisfy the symmetry condition at $u = 0$, the first integration constant must be set to $C_1 = 0$. The second boundary condition leads to the relationship

$$-\frac{Sb}{k(n+1)} = -\frac{h}{k}\left[-\frac{Sb^2}{2k(n+1)} + C_2 - T_f\right]$$

The previous relationship gives the second constant of integration

$$C_2 = T_f + \frac{Sb}{h(n+1)} + \frac{Sb^2}{2k(n+1)}$$

General Temperature Distribution

The general temperature distribution can be written as

$$T(u) - T_f = \frac{S}{2k(n+1)}(b^2 - u^2) + \frac{Sb}{h(n+1)}, \qquad 0 \le u \le b$$

which is valid for $n = 0$ (plane wall), $n = 1$ (long solid cylinder), and $n = 2$ (solid sphere).

Wall or Surface Temperature Drop

The temperature drop from the wall (or surface) to the fluid is obtained from the general solution by setting $u = b$ and $T(b) = T_s$. Therefore we get

$$T_s - T_f = \frac{Sb}{h(n+1)}$$

Centerline or Axis Temperature Drop

The temperature drop from the centerline or the axis where the maximum temperature occurs to the fluid is obtained from the general solution by setting $u = 0$ and $T(0) = T_{\max}$. Therefore we get

$$T_{\max} - T_f = \frac{Sb^2}{2k(n+1)} + \frac{Sb}{h(n+1)}$$

Solid Temperature Drop

The temperature drop across the solid is

$$T_{\max} - T_{\mathrm{s}} = \frac{Sb^2}{2k(n+1)}$$

Ratio of Solid to Film Temperature Drops

The ratio of the solid temperature drop to the film temperature drop is obtained from

$$\frac{\Delta T_{\mathrm{solid}}}{\Delta T_{\mathrm{film}}} = \frac{T_{\max} - T_{\mathrm{s}}}{T_{\mathrm{s}} - T_f} = \frac{1}{2}\frac{hb}{k} = \frac{1}{2}Bi$$

where Bi is the Biot number.