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Fins or Extended Surfaces

Fins or extended surfaces are used to increase the heat transfer rate from surfaces which are convectively cooled by gases (air) under natural or forced convection. The characteristics of fins: (a) they are metallic, (b) they having different shapes, (c) the fin length is much larger than the thickness or diameter, (d) there is perfect or imperfect contact at the base, (e) the fin tip is adiabatic or it is cooled, (f) the temperature distribution is one-dimensional because $Bi < 0.2$.

Derivation of Fin Equation

The derivation of the fin equation is based on a heat balance over the boundaries of a differential volume $dV = A(x)\,dx$ where $A(x)$ is the variable conduction area. The heat conduction rate into the volume through the boundary located at x according to Fourier's Law of Conduction is:

$$\dot{Q}_x = -kA(x)\,\frac{d\theta(x)}{dx}$$

where $\theta(x) = T(x) - T_f$ is the local temperature excess. The heat conduction rate out of the control volume through the boundary located at $x + dx$ is

$$\dot{Q}_{x+dx} = \dot{Q}_x + \frac{d}{dx}\dot{Q}_x\,dx + \text{higher order terms of Taylor series expansion}$$

The heat loss rate from the surface of the fin by convective cooling according to Newton's Law of cooling is

$$\dot{Q}_{\text{loss}} = h\,P(x)\,\theta(x)\,dx$$

where h is the uniform heat transfer coefficient and $P(x)$ is the local fin perimeter. For steady-state and in the absence of thermal sources or sinks, the energy balance over the boundaries of the control volume, ie:

$$\dot{Q}_x = \dot{Q}_{x+dx} + \dot{Q}_{\text{loss}}$$

leads to the following energy balance:

$$\frac{d}{dx}\dot{Q}_x\,dx + h\,P(x)\,\theta(x)\,dx = 0$$

General Fin Equation

Assuming the thermal conductivity to be constant, and after some manipulations the general fin equation is obtained:

$$\frac{d^2\theta}{dx^2} + \frac{1}{A(x)}\frac{dA(x)}{dx}\frac{d\theta(x)}{dx} - \frac{h}{k}\frac{P(x)}{A(x)}\theta(x) = 0, \qquad 0 \leq x \leq L$$

where L is the length of the fin. The above equation is second-order with variable coefficients. It requires two boundary conditions: (i) at the fin base ($x = 0$) and (ii) at the fin tip ($x = L$).

Boundary Conditions

At the fin base $x = 0$ there are two possible conditions: (a) perfect contact where $T(0) = T_b$ which requires that $\theta(0) = \theta_{\text{b}} = T_{\text{b}} - T_f$ and T_{b} is the base temperature, or there is imperfect contact at the base in which case we have $q_b = h_{\text{c}}\,[T_{\text{b}} - T(0)] = -k\ dT/dx$ where h_{c} is the contact conductance. The selection of the imperfect contact case leads to the boundary condition of the third kind:

$$\frac{d\theta(0)}{dx} = -\frac{h_{\text{c}}}{k}\,[\theta_{\text{b}} - \theta(0)]$$

At the fin tip $x = L$ there are three possible conditions: (a) adiabatic (insulated) tip where $dT/dx = d\theta/dx = 0$, (b) perfect contact with the fluid where $T(L) = T_f$, therefore $\theta(L) = 0$, and (c) convective cooling at the fin tip such that $q_{\text{tip}} = h_{\text{e}}\,[T(L) - T_f] = -k\ dT(L)/dx$ where h_{e} is the convective coefficient. The third case will be considered here because it leads to the general fin solution. Therefore at the fin tip we take:

$$\frac{d\theta(L)}{dx} = -\frac{h_{\text{e}}}{k}\theta(L)$$

Fin Equation for Constant Cross-Sections

For constant cross-section fins: $A(x) = A$ and $P(x) = P$, therefore the general fin equation becomes:

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0, \qquad 0 \leq x \leq L$$

where the fin parameter is defined as

$$m^2 = \frac{hP}{kA}$$

and its units are m^{-2}. The hyperbolic form of the solution of the previous second-order differential equation is chosen:

$$\theta = C_1 \cosh mx + C_2 \sinh mx$$

The temperature gradient is

$$\frac{d\theta}{dx} = m\, C_1 \sinh mx + m\, C_2 \cosh mx$$

Dimensionless Fin Parameters

We introduce the following three dimensionless fin parameters which account for heat transfer through the base, the fin tip and the fin sides:

$$Bi_{\rm c} = \frac{h_{\rm c} L}{k}, \quad Bi_{\rm e} = \frac{h_{\rm e} L}{k}, \quad mL = \sqrt{\tfrac{hP}{kA}}\, L, \quad Bi = \frac{h t_{\rm e}}{k} < 0.2$$

where $t_{\rm e}$ is some *effective* thickness of the fin cross-section. The effective thickness of a rectangular cross-section: $2t$ by w where w is the width and $w >> 2t$ is $t_{\rm e} = t$. This relation is consistent with the definition:

$$t_{\rm e} = \frac{A}{P} = \frac{\rm cross - sectional\ area}{\rm perimeter}$$

For a circular fin of diameter d, $t_{\rm e} = A/P = (\pi/4d^2)/(\pi d) = d/4$; for a fin of square cross-section where $A = 4w^2$, $t_{\rm e} = A/P = (4w^2)/(8w) = w/2$.

Constants of Integration

After some algebraic manipulations the constants of integration for the two boundary conditions of the third kind give:

$$C_1 = \theta_{\rm b} \left[1 + \frac{mL\phi}{Bi_{\rm c}} \right]^{-1} \qquad [K]$$

and

$$C_2 = -\theta_{\rm b}\, \phi \left[1 + \frac{mL\phi}{Bi_{\rm c}} \right]^{-1} \qquad [K]$$

The fin function ϕ is defined as

$$\phi = \frac{mL \tanh mL + Bi_{\rm e}}{mL + Bi_{\rm e} \tanh mL}$$

Fin Heat Transfer Rate

The relation for the heat transfer rate through the fin can be obtained by the application of Fourier's Law of Conduction at the fin base:

$$\dot{Q}_{\text{fin}} = -kA_{\text{b}}\frac{d\theta(0)}{dx} = -k\,A\,m\,C_2 \;\; [W]$$

Fin Resistance

The fin resistance is defined as:

$$R_{\text{fin}} = \frac{T_{\text{b}} - T_f}{\dot{Q}_{\text{fin}}} = \left[1 + \frac{mL\,\phi}{Bi_{\text{c}}}\right]\left[\sqrt{hPkA}\,\phi\right]^{-1}$$

The general solution and corresponding relations can be used for any constant cross-section fin which has contact resistance and end cooling. The special cases which are frequently presented in heat transfer texts arise from the previous general solution and results.

Special Cases of the General Solution

Perfect Contact at the Fin Base and End Cooling

For this case we put $h_{\text{c}} \geq 10^9$ or $Bi_{\text{c}} \geq 10^9$. This leads to $\theta(0) = \theta_{\text{b}}$ or $T(0) = T_{\text{b}}$. The fin resistance becomes:

$$R_{\text{fin}} = \frac{1}{\sqrt{hPkA}\,\phi}$$

where the fin function $\phi = \phi(mL, Bi_{\text{e}})$. When $Bi_{\text{e}} = 0$, $\phi = \tanh mL$, and when $Bi_{\text{e}} = \infty$, $\phi = \coth mL$. Also when $mL \geq 2.65$, the numerical values of $\tanh mL$ and $\coth mL$ are within 1% of 1, and therefore $\phi \approx 1$ for all values of Bi_{e}.

Perfect Contact at Fin Base and Adiabatic Fin Tip

For this case we put $Bi_{\text{c}} = 10^9$ and $Bi_{\text{e}} = 0$. The fin function becomes: $\phi = \tanh mL$ and the fin resistance relation becomes:

$$R_{\text{fin}} = \frac{1}{\sqrt{hPkA}\,\tanh mL}$$

Infinitely Long Fin With Perfect Contact at Fin Base

For this case we put $Bi_c = 10^9$ and take $mL \geq 2.65$, and the fin resistance reduces to

$$R_{\text{fin}} = \frac{1}{\sqrt{hPkA}}$$

Criterion for Infinitely Long Fins

The criterion for infinitely long fins is

$$L_{\text{infinitely long}} \geq 2.65\sqrt{\frac{kA}{hP}}$$

Temperature Distributions for the Special Cases

Perfect Contact at the Fin Base and Tip Cooling

$$\frac{\theta(x)}{\theta_{\text{b}}} = \cosh mx - \phi \sin mx, \qquad 0 \leq x \leq L$$

where ϕ is defined above.

Perfect Contact at Fin Base and Fin Tip

$$\frac{\theta(x)}{\theta_{\text{b}}} = \frac{\sinh m(L-x)}{\sinh mL}, \qquad 0 \leq x \leq L$$

Perfect Contact at Fin Base and Adiabatic Fin Tip

$$\frac{\theta(x)}{\theta_{\text{b}}} = \frac{\cosh m(L-x)}{\cosh mL}, \qquad 0 \leq x \leq L$$

Infinitely Long Fin With Perfect Contact at Base

$$\frac{\theta(x)}{\theta_{\text{b}}} = e^{-mx}, \qquad 0 \leq x \leq L \geq 2.65\sqrt{\frac{kA}{hP}}$$

Fin Efficiency

The fin efficiency is defined for fins with perfect base contact and adiabatic tip as:

$$\eta = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{ideal}}} < 1$$

where the ideal fin heat transfer rate is defined as

$$\dot{Q}_{\mathrm{ideal}} = \int_0^L h\,P\,\theta\,dx$$

which becomes

$$\dot{Q}_{\mathrm{ideal}} = h\,P\,L\,\theta_{\mathrm{b}}$$

when $\theta(x) = \theta_{\mathrm{b}}$ which corresponds to fins whose thermal conductivity approaches infinitely large values.