ECE 309 M.M. Yovanovich

Week 2

Lecture 1

Review Stefan-Boltzmann Law of Radiative exchange between two isothermal convex gray surfaces: A_1, ϵ_1, T_1 and A_2, ϵ_2, T_2.

- **Radiative Film Coefficient**

- Definition

$$\dot{Q}_{\rm rad} = h_r A_1 (T_1 - T_2)$$

Equate to

$$\dot{Q}_{12} = \frac{\sigma(T_1^2 + T_2^2)(T_1 + T_2)(T_1 - T_2)}{R_{\rm rad}}$$

to get

$$h_r = \frac{\sigma(T_1^2 + T_2^2)(T_1 + T_2)}{A_1 R_{\rm rad}}$$

The radiative film coefficient is a nonlinear complex system parameter:

$$h_r = f(\sigma, T_1, T_2, A_1, A_2, \epsilon_1, \epsilon_2, F_{12})$$

- Multimode Heat Transfer Example.

Steady heat transfer through a plane wall: (k, L, A) with boundary temperatures: (T_1, T_2) where $T_1 > T_2$. The heat transfer through the wall $\dot{Q}_{\rm cond}$ leaves the right boundary at temperature T_2 and splits into two streams: $\dot{Q}_{\rm conv}, \dot{Q}_{\rm rad}$ and some of the heat goes to the fluid at temperature T_∞ by convective heat transfer, and the remainder goes to the surroundings at temperature $T_{\rm sur}$ by radiative heat transfer.

$$\dot{Q}_{\rm cond} = \dot{Q}_{\rm conv} + \dot{Q}_{\rm rad}$$

by conservation of energy principle. Also we have the relations:

$$\dot{Q}_{\rm cond} = \frac{T_1 - T_2}{R_{\rm cond}}, \quad \dot{Q}_{\rm conv} = hA(T_2 - T_\infty), \quad \dot{Q}_{\rm rad} = h_r A(T_2 - T_{\rm sur})$$

where $R_{\text{cond}} = L/(kA)$.

• Equivalent thermal circuit with temperature nodes: $T_1, T_2, T_\infty, T_{\text{sur}}$, three thermal resistors: $R_{\text{cond}}, R_{\text{conv}}, R_{\text{rad}}$, and three throughputs: $\dot{Q}_{\text{cond}}, \dot{Q}_{\text{conv}}, \dot{Q}_{\text{rad}}$

Lecture 2

• **Fourier's Law of Conduction**

$$\vec{q} = -k\nabla T$$

• Heat flux vector and temperature gradient in cartesian coordinates:

$$\vec{q} = \vec{i}q_x + \vec{j}q_y + \vec{k}q_z \quad \text{and} \quad \nabla T = \vec{i}\,\frac{\partial T}{\partial x} + \vec{j}\,\frac{\partial T}{\partial y} + \vec{k}\,\frac{\partial T}{\partial z}$$

and the unit vectors are: $\vec{i}, \vec{j}, \vec{k}$, respectively.

• Heat flux components in cartesian coordinates (x, y, z):

$$q_x = -k\frac{\partial T}{\partial x}, \qquad q_y = -k\frac{\partial T}{\partial y}, \qquad q_z = -k\frac{\partial T}{\partial z}$$

Similarly,
• Heat flux components in cylindrical coordinates (r, θ, z):

$$q_r = -k\frac{\partial T}{\partial r}, \qquad q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta}, \qquad q_z = -k\frac{\partial T}{\partial z}$$

• Heat flux components in spherical coordinates (r, θ, ϕ):

$$q_r = -k\frac{\partial T}{\partial r}, \qquad q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta}, \qquad q_\phi = -k\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

We will consider the one-dimensional conduction problems where $T(x)$ and $T(r)$.

• Steady conduction in plane wall: (A, L, k) without heat sources. The boundary temperatures are: i) $x = 0, T = T_1$ and ii) $x = L, T = T_2 < T_1$.
Conservation of energy principle applied to a differential control volume $dV = Adx$ gives

$$\frac{dQ_x}{dx}dx = \frac{d}{dx}\left[-k(T)A\frac{dT}{dx}\right]dx = 0$$

Dividing by the differential volume gives

$$\frac{d}{dx}\left[-k(T)\frac{dT}{dx}\right] = 0, \qquad 0 < x < L$$

for the plane wall with temperature dependent thermal conductivity: $k(T)$. Integrating once gives

$$k(T)\frac{dT}{dx} = C_1$$

The second integration gives

$$\int_{T_1}^{T_2} k(T)dT = C_1 L$$

Left hand side can be written as

$$-\frac{(T_1 - T_2)}{(T_1 - T_2)}\int_{T_1}^{T_2} k(T)dT = -k_{ave}(T_1 - T_2)$$

where the average value of the thermal conductivity in the temperature interval $[T_2, T_1]$ is defined as

$$k_{ave} = \frac{1}{(T_1 - T_2)}\int_{T_1}^{T_2} k(T)dT$$

The constant of integration is

$$C_1 = -k_{ave}\frac{(T_1 - T_2)}{L}$$

To obtain the temperature distribution if $k(T)$ requires a particular relation for the thermal conductivity as a function of temperature. Frequently a linear relation is used for a small temperature range.

When $k(T) = k$, a constant, then the equation becomes

$$\frac{d^2T}{dx^2} = 0, \qquad 0 < x < L$$

Two integrations give

$$T(x) = C_1 x + C_2$$

Applying the boundary conditions to find the constants of integration gives the linear temperature distribution:

$$T(x) = T_1 - \frac{x}{L}(T_1 - T_2), \qquad 0 \leq x \leq L$$

- Heat flow rate, thermal resistance and shape factor.

$$Q_x = -kA\frac{dT}{dx} = kA\frac{(T_1 - T_2)}{L}$$

$$R = \frac{(T_1 - T_2)}{Q_x} = \frac{L}{kA}, \qquad S = \frac{1}{kR} = \frac{A}{L}$$

Lecture 3

• Notation used in text: $q^{'}, q^{''}, q^{'''}$ denote heat transfer rate per unit length, per unit area, and per unit volume respectively.

• Steady Conduction in Hollow Cylinder: $a \leq r \leq b << L$ where L is the cylinder length. The thermal conductivity of the cylinder wall is $k = k(T)$. The inner and outer boundaries are maintained at uniform temperatures:
$T(r = a) = T_1, T(r = b) = T_2 < T_1$. The temperature in the cylinder wall is one-dimensional, i.e., $T(r)$.

• Fourier's Law of Conduction gives the heat flow rate at the inner surface of the differential control volume CV located at r:

$$Q_r = q_r A_r = -k(T) 2\pi r L \frac{dT}{dr}$$

where the conduction area is $A_r = 2\pi r L$.
The heat flow rate out of the CV through the surface at $r + dr$ is by Taylor series expansion of Q_r:

$$Q_r + \frac{dQ_r}{dr} dr + \text{HOTTSE}$$

If distributed volumetric heat sources are not present and the conduction is time independent, the conservation of energy principle states that the conduction rate into the CV must equal the conduction rate out of the CV. Therefore

$$Q_r = Q_r + \frac{d}{dr}\left(-k(T) 2\pi r L \frac{dT}{dr}\right) dr$$

Cancelling the Q_r terms and dividing through by the control volume $dV = 2\pi r L dr$ gives the ordinary differential equation:

$$\frac{1}{r}\frac{d}{dr}\left(k(T)\, r \frac{dT}{dr}\right) = 0, \qquad a < r < b$$

The boundary conditions are

$$r = a, \quad T = T_1 \qquad \text{and} \qquad r = b, \quad T = T_2 < T_1$$

The ODE cannot be integrated for arbitrary $k(T)$. We will assume that $k =$ constant. Remove k and divide through by k to get the ODE:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0, \qquad a < r < b$$

which can be integrated twice:

$$r\frac{dT}{dr} = C_1 \qquad \text{or} \qquad \frac{dT}{dr} = \frac{C_1}{r}$$

and

$$T(r) = C_1 \ln r + C_2$$

Use the boundary conditions to get two relations:

$$T_1 = C_1 \ln a + C_2 \qquad \text{and} \qquad T_2 = C_1 \ln b + C_2$$

Substracting to eliminate C_2 gives for C_1

$$-C_1 = \frac{(T_1 - T_2)}{\ln \frac{b}{a}}$$

¿From either relation C_2 can be found. Taking the first relation gives:

$$C_2 = T_1 - C_1 \ln a$$

Substituting into the general solution gives the temperature distribution in the cylinder wall:

$$\frac{T_1 - T(r)}{T_1 - T_2} = \frac{\ln \frac{r}{a}}{\ln \frac{b}{a}}, \qquad a \leq r \leq b$$

- Heat flow rate through cylinder wall.

$$Q_r = -k2\pi r L\frac{dT}{dr} = -k2\pi r L\frac{C_1}{r} = -C_1 2\pi L k = 2\pi L k\frac{(T_1 - T_2)}{\ln \frac{b}{a}}$$

- Thermal resistance and shape factors

$$R = \frac{(T_1 - T_2)}{Q_r} = \frac{1}{2\pi L k}\ln \frac{b}{a}$$

and

$$S = \frac{1}{kR} = \frac{2\pi L}{\ln \dfrac{b}{a}}$$

These are important results which will be used to obtain resistances of systems.

- Steady Conduction through Wall of Hollow Sphere.

- Outline of the procedure which is similar to that of the hollow cylinder.

- Fourier's Law of Conduction in a Sphere where $T(r)$:

$$Q_r = q_r A_r = -k4\pi r^2 \frac{dT}{dr}$$

- Apply conservation of energy principle on differential CV where $dV = 4\pi r^2 dr$ to get the ODE:

$$\frac{1}{r^2}\frac{d}{dr}\left(k(T) r^2 \frac{dT}{dr}\right) = 0, \qquad a < r < b$$

- Boundary conditions are

$$r = a, \quad T = T_1 \qquad \text{and} \qquad r = b, \quad T = T_2 < T_1$$

- Integrate twice to get the general solution of ODE if the thermal conductivity is constant. ODE becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0, \qquad a < r < b$$

- The general solution is

$$T(r) = \frac{C_1}{r} + C_2$$

- Temperature distribution is

$$\frac{T_1 - T(r)}{T_1 - T_2} = \frac{1/a - 1/r}{1/a - 1/b}, \qquad a \le r \le b$$

- Heat flow rate through spherical wall.

$$Q_r = -k4\pi r^2 \frac{dT}{dr} = 4\pi k (T_1 - T_2) \frac{1}{1/a - 1/b}$$

• Thermal resistance and shape factor of the spherical wall.

$$R = \frac{(T_1 - T_2)}{Q_r} = \frac{1}{4\pi k}\left(\frac{1}{a} - \frac{1}{b}\right)$$

and

$$S = \frac{1}{kR} = \frac{4\pi}{1/a - 1/b}$$

• Isolated Sphere
Let $b/a \to \infty$ to get results for isolated sphere in an infinite substance of thermal conductivity k.

$$R = \frac{1}{4\pi a k} \qquad \text{and} \qquad S = 4\pi a$$

• Capacitance of an Isolated Sphere.

$$C = 4\pi a \epsilon$$

where ϵ is the permittivity of space.

• Electrical and Thermal Analogs
Observe that the electrical capacitance is related to the thermal resistance and shape factor

$$\frac{C}{\epsilon} = 4\pi a = S = \frac{1}{kR}$$
