
Week 8

Lecture 1
Provide midterm results. Problem 3 of midterm will be re-submitted at start of next lecture.
Outline of the solution procedure to be followed.

Lecture 2
Hand in Problem 3.
Half-space solutions.
Neumann Solution:

$$T(x,t) = T_i + \frac{q_0}{k}\left[\frac{2}{\sqrt{\pi}}\sqrt{\alpha t}\, e^{-x^2/4\alpha t} - x\, erfc\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

Instantaneous surface temperature rise: $T(0,t) = T_s$.

$$T_s = T_i + \frac{2}{\sqrt{\pi}}\frac{q_0}{k}\sqrt{\alpha t}$$

Robin Solution:

$$\frac{T(x,t) - T_i}{T_f - T_i} = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left[\frac{hx}{k} + \left(\frac{h}{k}\right)^2 \alpha t\right] erfc\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

Instantaneous surface temperature rise: $T(0,t) = T_s$.

$$\frac{T_s - T_i}{T_f - T_i} = 1 - \exp\left[\left(\frac{h}{k}\right)^2 \alpha t\right] erfc\left(\frac{h\sqrt{\alpha t}}{k}\right)$$

Instantaneous surface heat flux: q_s

$$q_s = -k\left(\frac{\partial T(x,t)}{\partial x}\right)_{x=0} = h(T_f - T_i)\, \exp\left[\left(\frac{h}{k}\right)^2 \alpha t\right]\, erfc\left(\frac{h\sqrt{\alpha t}}{k}\right)$$

Approximations of error and complementary error functions from P. R. Greene, *J. Fluids Engineering*, Vol. 111, pp. 224-226.

$$erf(x) = 1 - A\exp\left[-B\left(x + C\right)^2\right]$$

and

$$erfc(x) = 1 - erf(x) = A\exp\left[-B\left(x + C\right)^2\right]$$

with coefficients:

$$A = 1.5577, \qquad B = 0.7182, \qquad C = 0.7856$$

Greene claims the approximations are accurate to 0.42%. This is acceptable accuracy for many engineering calculations.

Inverse of Complementary Error Function
Inverse of $y = erfc(x)$ is $x = erfc^{-1}(y)$ where $0 \le y < 1$ and $x \ge 0$.

$$x = -C + \sqrt{-\frac{1}{B}\ln\left(\frac{y}{A}\right)}$$

Accuracy of inverse is unknown.

Commence overview of 1D transient solutions in plane wall, long circular cylinder, and solid sphere. See sections 5.4 through 5.6. Discuss temperature variation in plane wall during cooling.

Lecture 3
Return Midterm Exam at end of lecture. Provide new statistics.

Internal transient conduction: plane wall $T(x,t)$, long solid circular cylinder $T(r,t)$ and solid sphere $T(r,t)$; See text and Web site for Maple worksheets for Heisler cooling charts for one-dimensional conduction.
Dimensionless temperature: $\phi(\zeta, Fo)$ depends on dimensionless position: $\zeta = x/L$ for wall of thickness: $2L$; $\zeta = r/a$ for cylinder and sphere of radius a; and dimensionless time: $Fo = \alpha t/\mathcal{L}^2$ where $\mathcal{L} = L$ or a.
Dimensionless temperature is defined as:

$$\phi_h = \frac{T_f - T(\zeta, Fo)}{T_f - T_i}, \qquad \text{for heating}$$

and

$$\phi_c = \frac{T(\zeta, Fo) - T_f}{T_i - T_f}, \qquad \text{for cooling}$$

General Form of Temperature Solutions

$$\phi(\zeta, Fo) = \sum_{n=1}^{\infty} A_n \exp\left(-\delta_n^2 Fo\right) S(\delta_n \zeta) \qquad Fo > 0$$

where A_n are the temperature Fourier coefficients; $S(\delta_n \zeta)$ is the space-dependent function, and $\exp(-\delta_n^2 Fo)$ is the time-dependent function.
The eigenvalues: δ_n are the positive, real roots of the characteristic equations:

$$\delta_n \sin \delta_n = Bi \cos \delta_n, \qquad \text{plane wall}$$

and

$$\delta_n J_1(\delta_n) = Bi J_0(\delta_n), \qquad \text{long cylinder}$$

where $J_0(x)$ and $J_1(x)$ are Bessel functions of the first kind of order 0 and 1, respectively; and

$$\delta_n \cos \delta_n = (1 - Bi) \sin \delta_n, \qquad \text{sphere}$$

where the Biot number: $Bi = h\mathcal{L}/k$ lies in the range $0 < Bi < \infty$.
Space-dependent Functions

$$S(\delta_n \zeta) = \cos(\delta_n \zeta), \qquad \text{plane wall}$$

and

$$S(\delta_n \zeta) = J_0(\delta_n \zeta), \qquad \text{long cylinder}$$

and

$$S(\delta_n \zeta) = \frac{\sin(\delta_n \zeta)}{(\delta_n \zeta)}, \qquad \text{sphere}$$

Fourier Temperature Coefficients A_n are obtained from:

$$A_n = \frac{2 \sin \delta_n}{\delta_n + \sin \delta_n \cos \delta_n}, \qquad \text{plane wall}$$

and

$$A_n = \frac{2 J_1(\delta_n)}{\delta_n \left[J_0^2(\delta_n) + J_1^2(\delta_n)\right]}, \qquad \text{long cylinder}$$

and

$$A_n = \frac{2(\sin \delta_n - \delta_n \cos \delta_n)}{\delta_n - \sin \delta_n \cos \delta_n}, \qquad \text{sphere}$$

Heat Loss
The heat loss is defined as

$$Q_{loss} = E_i - E(t) = \rho c_P \theta_i V - \rho c_P \bar{\theta} V = \rho c_p V \left(\theta_i - \bar{\theta}\right)$$

where E_i and $E(t)$ represent the internal energy within the body initially and at arbitrary time $t > 0$, and V is the total volume of the body. The thermophysical properties are assumed to be constant during the cooling process. The volume average body temperature excess is defined as

$$\bar{\theta} = \frac{1}{V} \iiint_V \theta \, dV$$

Heat Loss Fraction

The heat loss fraction is defined as

$$\frac{E_i - E(t)}{E_i}, \qquad t > 0$$

which gives

$$\frac{Q}{Q_i} = 1 - \frac{\bar{\theta}}{\theta_i} = 1 - \sum_{n=1}^{\infty} B_n \exp\left(-\delta_n^2 Fo\right), \qquad Fo > 0$$

where $Q_i = E_i$ for convenience. The heat loss fraction Fourier coefficients B_n are obtained from:

$$B_n = A_n \frac{\sin \delta_n}{\delta_n}, \qquad \text{plane wall}$$

and

$$B_n = 2A_n \frac{J_1(\delta_n)}{\delta_n} = \frac{4Bi^2}{\delta_n^2(\delta_n^2 + Bi^2)}, \qquad \text{long cylinder}$$

and

$$B_n = \frac{6Bi^2}{\delta_n^2(\delta_n^2 + Bi^2 - Bi)}, \qquad \text{sphere}$$

Long Time Solutions

For $Fo > Fo_c$ where $Fo_c = 0.24, 0.21, 0.18$ for plane wall, long cylinder and solid sphere, respectively, the general solution converges to the first term of the summation, i.e.,

$$\phi(\zeta, Fo) = A_1 \exp(-\delta_1^2 Fo) S(\delta_1 \zeta)$$

and the first eigenvalue can be approximated by the correlation equation:

$$\delta_1 = \delta_{1,\infty} \left[1 + \left(\frac{\delta_{1,\infty}}{\delta_{1,0}}\right)^n\right]^{-1/n}$$

where $\delta_{1,0}$ is the asymptotic value as $Bi \to 0$ and $\delta_{1,\infty}$ is the asymptotic value as $Bi \to \infty$. The correlation coefficients are given in the following table:

Geometry	Wall	Cylinder	Sphere
$\delta_{1,0}$	$\sqrt{Bi}$	$\sqrt{2Bi}$	$\sqrt{3Bi}$
$\delta_{1,\infty}$	$\pi/2$	2.4048255	π
n	2.139	2.238	2.314

Heat Loss Fraction for Long Time

The series solution converges to the first term of the summation, and

$$\frac{Q(Fo)}{Q_i} = 1 - B_1 \exp(-\delta_1 Fo)$$

See ME 353 Web site for Maple worksheets for plane wall, long cylinder, and sphere. The worksheets show the full solutions and the first term approximations.

Lumped Capacitance Model

For all time $Fo > 0$ and sufficiently small Biot numbers, $Bi = h\mathcal{L}/k < 0.2$, where $\mathcal{L} = L$, a for the plane wall, and cylinder and sphere,

$$\frac{\theta(Fo)}{\theta_i} = e^{-BiFo}, \qquad \text{plane wall}$$

and

$$\frac{\theta(Fo)}{\theta_i} = e^{-2BiFo}, \qquad \text{long cylinder}$$

and

$$\frac{\theta(Fo)}{\theta_i} = e^{-3BiFo}, \qquad \text{sphere}$$

or, in general, for all systems:

$$\frac{\theta(t)}{\theta_i} = \exp\left(-\frac{hS}{\rho c_P V} t\right), \qquad t > 0$$

and

$$\frac{Q(t)}{Q_i} = 1 - \exp\left(-\frac{hS}{\rho c_P V} t\right), \qquad t > 0$$
