Week 7

Lecture 1
Read Chapter 5: Sections 5.1 - 5.8.
Lumped Capacitance Model (LCM)
$Bi = h\mathcal{L}/k < 0.2$; $T(\vec{r}, t) = T(t)$
System parameters:
$V, S, \rho, c_P, k, \alpha, h, T_i, T_f, T_{\rm surr}, q_{\rm in}, \dot{E}_{\rm gen}, \dot{E}_{\rm storage}, Q_{\rm conv}, Q_{\rm rad}$
Energy balance on control volume gives governing equation:

$$Q_{\rm in} + \dot{E}_{\rm gen} = \dot{E}_{\rm storage} + Q_{\rm conv} + Q_{\rm rad}$$

$Q_{\rm in} = q_{\rm in} S$; $\dot{E}_{\rm gen} = \mathcal{P} V$; $\dot{E}_{\rm storage} = \rho V c_P d\theta(t)/dt$ where $\theta(t) = T_f - T(t)$ for heating or $\theta(t) = T(t) - T_f$ for cooling; $Q_{\rm conv} = hS\theta$; $Q_{\rm rad} = \epsilon\sigma S(T^4(t) - T_{\rm surr}^4)$;
ode:

$$\frac{d\theta}{dt} = -\frac{h_0 S}{\rho c_P V}\,\theta + \frac{q_{\rm in} S}{\rho c_P V} + \frac{\mathcal{P}}{\rho c_P}$$

or for convenience:

$$\frac{d\theta}{dt} = -m\,\theta + n + p, \qquad t > 0$$

where $h_0 = h + h_{\rm rad}$ represents a combined linear heat transfer coefficient.
IC: $\theta(0) = \theta_i$
ODE is *nonlinear* when radiation is present, therefore numerical solutions are required.
Examine general and special cases:
(a) $q_{\rm in} > 0$ and $\mathcal{P} > 0$ or $n > 0$ and $p > 0$, general case
(b) $q_{\rm in} > 0$ and $\mathcal{P} = 0$ or $n > 0$ and $p = 0$
(c) $q_{\rm in} = 0$ and $\mathcal{P} > 0$ or $n = 0$ and $p > 0$
(d) $q_{\rm in} = 0$ and $\mathcal{P} = 0$ or $n = 0$ and $p = 0$
Solutions for the general and special cases are respectively:

$$\theta(t) = \frac{n}{m} + \frac{p}{m} + \left[\theta_i - \frac{n}{m} - \frac{p}{m}\right] e^{-mt}$$

$$\theta(t) = \frac{n}{m} + \left[\theta_i - \frac{n}{m}\right] e^{-mt}$$

$$\theta(t) = \frac{p}{m} + \left[\theta_i - \frac{p}{m}\right] e^{-mt}$$

$$\theta(t) = \theta_i e^{-mt}$$

Cooling or heating duration for general case $n > 0, p > 0$:

$$t = -\frac{1}{m} \ln \left[\frac{\theta - \frac{n}{m} - \frac{p}{m}}{\theta_i - \frac{n}{m} - \frac{p}{m}} \right]$$

Steady-state solution: when $d\theta(t)/dt = 0$, then $\theta_{ss} = n/m + p/m$ for the general case.
Time constant or charactersitic time of the system is

$$t_c = \frac{1}{m} = \frac{\rho c_P V}{hS}$$

See Website and Maple worksheets
See text for examples and problem set with solutions.

Lecture 2
Return Project 1. Well done in general.
Characteristic body length in Biot number: $Bi = h\mathcal{L}/k$; $\mathcal{L} = V/S$;
discuss LCM with Ohmic (Joulean) heating;

Lecture 3
One-dimensional transient conduction $T(x,t)$ in halfspace $x > 0$:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \qquad t > 0$$

Initial condition: $t = 0, T(x,0) = T_i$, constant for $x \geq 0$
Boundary condition as $x \to \infty$ is $T(x,t) \to T_i$

Three types of boundary conditions at free surface $x = 0$:
(i) BC of First Kind or Dirichlet condition: $T(0,t) = T_0 > T_i$
(ii) BC of Second Kind or Neumann condition: $\partial T(x,t)/\partial x = -q_0/k$
(iii) BC of the Third Kind or Robin condition: $\partial T(x,t)/\partial x = -h/k\,[T_f - T(0,t)]$
Introduce temperature excess: $\theta(x,t) = T(x,t) - T_i$ for heating problems and $\theta(x,t) = T_i - T(x,t)$ for cooling problems.
See text for solutions for Dirichlet, Neumann and Robin conditions.
See Website for Maple worksheets.

Solutions are obtained by means of Laplace Transform Methods. For Dirichlet problem, similarity parameter $\eta = x/\sqrt{4\alpha t}$, can be employed to transform PDE into ODE (see text, pages 237-238).

Dirichlet solution:

$$\frac{\theta}{\theta_i} = erfc\,(\eta) = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) \qquad t > 0$$

Instantaneous surface flux:

$$q(0,t) = -k\left[\frac{\partial T}{\partial x}\right]_{x=0} = -k\left[\frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x}\right]_{x=0,\eta=0} = \frac{2}{\sqrt{\pi}}\frac{(T_0 - T_i)}{\sqrt{\alpha t}}$$
