# LAPLACE TRANSFORM

The Laplace transform of $f(x,t)$ is defined as

$$\mathcal{L}\left\{f(x,t)\right\} = \int_0^\infty e^{-st}\, f(x,t)dt = F(x,s) \qquad s > 0$$

and its inverse is defined as

$$\mathcal{L}^{-1}\left\{F(x,s)\right\} = \frac{1}{2\pi i}\, \int_{c-i\infty}^{c+i\infty}\, F(x,s)e^{st}\, ds = f(x,t)$$

## Summary of Laplace Transforms of Partial Derivatives and Partial Differential Equations

1) **Constant**

$$\mathcal{L}\left\{c\right\} = \int_0^\infty e^{-st}\, c\; dt = c\int_0^\infty e^{-st}\; dt = \frac{c}{s} \qquad s > 0$$

$$\mathcal{L}^{-1}\left\{\frac{c}{s}\right\} = c\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = c$$

2) **Spatial Partial Derivatives**

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \int_0^\infty e^{-st}\, \frac{\partial u}{\partial x}\; dt = \frac{d}{dx}\int_0^\infty e^{-st}\, u dt$$

by Leibnitz's Rule

Therefore

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{d}{dx}\mathcal{L}\left\{u\right\} \;\equiv \frac{d\overline{u}}{dx}(x,s)$$

Similarly,

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^\infty e^{-st}\, \frac{\partial^2 u}{\partial x^2}\; dt = \frac{d^2}{dx^2}\; \int_0^\infty e^{-st}\; u\; dt = \frac{d^2\overline{u}}{dx^2}\,(x,s)$$

3) **Temporal Partial Derivatives**

The first partial derivative of $u$ with respect to the temporal (time) variable $t$ can be transformed by application of the above definition.

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = \int_0^\infty e^{-st}\frac{\partial u}{\partial t}\, dt$$

Integrating by parts with

$$\begin{array}{ll} u' = e^{-st} & dv' = du \\ du' = -se^{-st}dt & v' = u \end{array}$$

we obtain:

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} &= [u'v']_0^\infty - \int_0^\infty u\frac{d}{dt}\, e^{-st} \\ &= [ue^{-st}]_0^\infty + s\int_0^\infty e^{-st}\, u\, dt \\ &= s\int_0^\infty e^{-st}\, u\, dt - u(x,0) \end{aligned}$$

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\overline{u}(x,s) - u(x,0)$$

The second partial derivative of $u$ with respect to time $t$ can be transformed as well.

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \int_0^\infty e^{-st}\frac{\partial^2 u}{\partial t^2}\, dt = \left[e^{-st}\frac{\partial u}{\partial t}\right]_0^\infty + s\int_0^\infty e^{-st}\,\frac{\partial u}{\partial t}\, dt$$

Therefore,

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} - \frac{\partial u}{\partial t}(x,0)$$

or

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s\left[s\overline{u}(x,s) - u(x,0)\right] - \frac{\partial u}{\partial t}(x,0)$$

Finally we have,

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\overline{u}(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0)$$

**Application to Diffusion Equation** $\nabla^2 u = \frac{1}{\alpha}\frac{\partial u}{\partial t}$

**PDE** $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$ $\qquad x \geq 0 \quad t > 0$

$\alpha > 0$

**IC** $u(x, 0) = 0$

**BCs** $u(0, t) = u_0$, constant

$u(x \to \infty, t) \to 0$

**Laplace Transformed Equation, IC and BCs**

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2} - \frac{1}{\alpha}\frac{\partial u}{\partial t}\right\} = \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} - \frac{1}{\alpha}\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = 0$$

$$= \frac{d^2\overline{u}}{dx^2} - \frac{1}{\alpha}[s\overline{u} - u(x, 0)] = 0$$

Therefore,

$$\frac{d^2\overline{u}}{dx^2} - \frac{s}{\alpha}\overline{u} = \frac{1}{\alpha}\, u(x, 0)$$

Since the initial condition requires $u(x, 0) = 0$, therefore,

$$\frac{d^2\overline{u}}{dx^2} - \frac{s}{\alpha}\,\overline{u} = 0 \qquad s > 0$$

**Laplace Transform of Boundary Conditions**

$$\mathcal{L}\{u(0, t)\} = \overline{u}(0, s) = \mathcal{L}\{u_0\} = \frac{u_0}{s}$$

and

$$\mathcal{L}\{u(x \to \infty, t)\} = \overline{u}(x \to \infty, s) = \mathcal{L}\{0\} = 0$$

**Solution in Transform Domain,** $s$

$$\overline{u} = C_1 e^{\sqrt{s/\alpha}\, x} + C_2 e^{-\sqrt{s/\alpha}\, x}$$

Because

$$\overline{u}(x,s) \to 0 \quad \text{as} \quad x \to \infty, \quad \text{therefore} \;\; C_1 = 0 \;.$$

Thus,

$$\overline{u}(x,s) = C_2 e^{-\sqrt{s/\alpha}\,x}$$

Applying the second boundary condition gives $\overline{u}(0,s) = C_2 = \dfrac{u_0}{s}$ .
The solution in the transform domain is

$$\overline{u}(x,s) = u_0 \frac{e^{-\sqrt{s/\alpha}\,x}}{s}$$

To find $u(x,t)$ we must take the *inverse* Laplace transform of the above solution, i.e., $\mathcal{L}^{-1}\left\{\overline{u}(x,s)\right\}$.
Therefore,

$$u(x,t) = u_0 \mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s/\alpha}\,x}}{s}\right\}$$

From Laplace Transform Tables we obtain

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = erfc\left(\frac{a}{2\sqrt{t}}\right)$$

In this problem we have $\quad a = \dfrac{x}{\sqrt{\alpha}}$

Therefore,

$$\begin{aligned} u(x,t) &= u_0 erfc(\frac{x}{2\sqrt{\alpha t}}) \\ &= u_0 \left[1 - erf(\frac{x}{2\sqrt{\alpha t}})\right] \\ &= u_0 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{x/2\sqrt{\alpha t}} e^{-\beta^2}\, d\beta\right] \end{aligned}$$

**Summary of Laplace Transform Solution of Diffusion Equation, BCs and IC**

| <u>$s$-Domain</u> | <u>$t$-Domain</u> |
|---|---|
| $\dfrac{d^2\overline{u}}{dx^2} - \dfrac{s}{\alpha}\overline{u} = 0$ | $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial u}{\partial t}$ |
| $\overline{u}(x,s) = u_0\,\dfrac{1}{s}e^{-\sqrt{s/\alpha}\,x}$ | $u(x,t) = u_0\; erfc\left(\dfrac{x}{2\sqrt{\alpha t}}\right)$ |
| At $x = 0$, | |
| $\overline{u}(0,s) = \dfrac{u_0}{s}$ | $u(0,t) = u_0$ |
| As $x \to \infty$, | |
| $\overline{u}(\infty,s) = 0$ | $u(\infty,t) = 0$ |

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**Laplace Transform of Heat Diffusion In a Finite Domain** $\quad 0 \le x \le a$

**PDE** $\quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial u}{\partial t} \qquad 0 < x < a, \;\; t > 0$

**IC** $\quad t = 0, \;\; 0 \le x \le a, \quad u(x,0) = 0$

**BCs** $\quad t > 0, \qquad u(0,t) = u_0,$ Constant

$$\frac{\partial u}{\partial x}(a,t) = 0$$

**Laplace Transform of PDE, IC and BCs**

**PDE** $\longrightarrow$ **ODE**

$$\frac{d^2\overline{u}}{dx^2} = \frac{1}{\alpha}\left[s\overline{u}(x,s) - u(x,0)\right] \quad \text{but } u(x,0) = 0$$

Therefore $\quad \dfrac{d^2\overline{u}}{dx^2} = \dfrac{s}{\alpha}\,\overline{u} \qquad 0 \le x \le a$

**BCs** $\quad \dfrac{d\overline{u}}{dx}(a,s) = 0 \qquad \overline{u}(0,s) = \dfrac{u_0}{s}$

**Solution in the Transform s-Domain**

We may use hyperbolic functions because the physical domain is finite, i.e., $0 \le x \le a$. Therefore the transformed solution is

$$\overline{u} = C_1 \cosh \sqrt{s/\alpha}\, x + C_2 \sinh \sqrt{s/\alpha}\, x$$

The first derivative of $\overline{u}(x, s)$ with respect to the spatial parameter $x$ is:

$$\frac{d\overline{u}}{dx} = C_1 \sqrt{s/\alpha}\ \sinh \sqrt{s/\alpha}\, x + C_2 \sqrt{s/\alpha}\ \cosh \sqrt{s/\alpha}\, x$$

Applying the boundary condition at $x = 0$ gives:

$$\overline{u}(0, s) = \frac{u_0}{s} = C_1$$

and the boundary condition at $x = a$ gives:

$$\frac{d\overline{u}}{dx}(a, s) = 0 = C_1 \sqrt{s/\alpha}\ \sinh \sqrt{s/\alpha}\, a + C_2 \sqrt{s/\alpha}\ \cosh \sqrt{s/\alpha}\, a$$

From the last equation we have $C_2 = -C_1 \dfrac{\sinh \sqrt{s/\alpha}\, a}{\cosh \sqrt{s/\alpha}\, a}$

Therefore,

$$\overline{u}(x, s) = C_1 \left[ \cosh \sqrt{s/\alpha}\, x - \frac{\sinh \sqrt{s/\alpha}\, a}{\cosh \sqrt{s/\alpha}\, a} \sinh \sqrt{s/\alpha}\, x \right] \qquad \text{or}$$

$$\overline{u}(x, s) = \frac{u_0}{s} \left[ \frac{\cosh \sqrt{s/\alpha}\, a \cosh \sqrt{s/\alpha}\, x - \sinh \sqrt{s/\alpha}\, a \sinh \sqrt{s/\alpha}\, x}{\cosh \sqrt{s/\alpha}\, a} \right]$$

Thus the solution in the transformed $s-$domain is

$$\overline{u}(x, s) = u_0 \left[ \frac{\cosh \sqrt{s/\alpha}\ (a - x)}{s \cosh \sqrt{s/\alpha}\, a} \right]$$

The solution in the physical domain $(x, t)$ is obtained by the *inverse* Laplace transform:

$$u(x, t) = \mathcal{L}^{-1} \left\{ \overline{u}(x, s) \right\} = u_0 \mathcal{L}^{-1} \left\{ \frac{\cosh \sqrt{s/\alpha}(a - x)}{s \cosh \sqrt{s/\alpha}\, a} \right\}$$

By means of Laplace Transform Tables (see Spiegel Page 171 #32.153) we have

$$\mathcal{L}^{-1}\left\{\frac{\cosh x\sqrt{s}}{s\cosh a\sqrt{s}}\right\} = 1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\, e^{-(2n-1)^2\frac{\pi^2 t}{4a^2}}\cos(2n-1)\frac{\pi}{2}\frac{x}{a}$$

Therefore the solution of the one-dimensional diffusion equation in a finite domain is

$$\frac{u(x,t)}{u_0} = 1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\, e^{-(2n-1)^2\frac{\pi^2}{4}\frac{\alpha t}{a^2}}\cos(2n-1)\frac{\pi}{2}\frac{(a-x)}{a}$$

Expanding the cosine function in the previous expression leads to the another equivalent form of the solution:

$$\frac{u(x,t)}{u_0} = 1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\, e^{-(2n-1)\frac{\pi^2}{4}\tau}\sin(2n-1)\frac{\pi}{2}\xi$$

with $\tau = \alpha t/a^2$ and $\xi = x/a$, the dimensionless time and dimensionless position respectively.
Expanding the first three terms of the summation gives:

$$\frac{u(\xi,t)}{u_0} = 1-\frac{4}{\pi}\left[e^{-\frac{\pi^2}{4}\tau}\cos\frac{\pi}{2}(1-\xi)-\frac{1}{3}\, e^{-\frac{9\pi^2}{4}\tau}\cos\frac{3\pi}{2}(1-\xi)\right.$$

$$\left.+\frac{1}{5}\, e^{-\frac{25\pi^2}{4}\tau}\cos\frac{5\pi}{2}(1-\xi)-\ldots\right]$$

Check convergence of solution at $\xi = \dfrac{x}{a} = 1$.

$$\frac{u(1,\tau)}{u_0} = 1-\frac{4}{\pi}\left[e^{-\frac{\pi^2}{4}\tau}-\frac{1}{3}\, e^{-9\frac{\pi^2}{4}\tau}+\frac{1}{5}\, e^{-25\frac{\pi^2}{4}\tau}-\ldots\right]$$

When $\tau = 0.1$ we find that

$$e^{-\frac{\pi^2}{4}(.1)} = 0.781344$$

$$\frac{1}{3}e^{-9\frac{\pi^2}{4}(.1)} = 0.036179$$

$$\frac{1}{5}e^{-25\frac{\pi^2}{4}(.1)} = 0.000419$$

For 3 decimal place accuracy we may use 2 terms because the higher order terms become negligible rapidly. More terms are required for short times, i.e., $\tau < 0.1$.

**Short Time Solution, $\tau \leq 0.1$**

A short time solution can be obtained by setting $m = \sqrt{\dfrac{s}{\alpha}}$ and $y = a - x$

Now, $u(y,t) = u_0 \mathcal{L}^{-1}\left\{\dfrac{\cosh my}{s \cosh ma}\right\}$

Examine

$$\frac{\cosh(my)}{\cosh(ma)} = \frac{e^{my} + e^{-my}}{e^{ma} + e^{-ma}} = \frac{e^{-ma}(e^{my} + e^{-my})}{1 + e^{-2ma}}$$

Therefore,

$$\begin{aligned}
\frac{\cosh(my)}{\cosh(ma)} &= \left[e^{-m(y-a)} + e^{-m(y+a)}\right] \sum_{k=0}^{\infty} (-1)^k e^{-2kma} \\
&= \sum_{n=0}^{\infty} (-1)^n \left\{\exp[-m(na-y)] + \exp[-m(na+y)]\right\}
\end{aligned}$$

where $n = 2k + 1$.

From Laplace Transform Tables we have

$$\mathcal{L}\left\{erfc(\frac{a}{2\sqrt{t}})\right\} = \frac{1}{s}\, e^{-a\sqrt{s}}$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{\cosh my}{s\cosh ma}\right\} = \sum_{k=0}^{\infty}(-1)^k\left\{erfc[\frac{(2k+1)a-y}{2\sqrt{\alpha t}}] + erfc[\frac{(2k+1)a+y}{2\sqrt{\alpha t}}]\right\}$$

The short time solution is

$$\frac{u(y,t)}{u_0} = 1-\sum_{k=0}^{\infty}(-1)^k \quad \left\{erfc\left[\frac{(2k+1)a-y}{2\sqrt{\alpha t}}\right] + erfc\left[\frac{(2k+1)a+y}{2\sqrt{\alpha t}}\right]\right\}$$

Let $\dfrac{\alpha t}{a^2}=\tau$ and $\dfrac{y}{a}=1-\dfrac{x}{a}=1-\xi$ , therefore

$$\begin{aligned}\frac{u(y/a,\tau)}{u_0} = 1 \; & -\left\{erfc\frac{1-y/a}{2\sqrt{\tau}}+erfc\,\frac{1+y/a}{2\sqrt{\tau}}\right\}\\ & +\left\{erfc\frac{3-y/a}{2\sqrt{\tau}}+erfc\frac{3+y/a}{2\sqrt{\tau}}\right\}\\ & -\left\{erfc\frac{5-y/a}{2\sqrt{\tau}}+erfc\frac{5+y/a}{2\sqrt{\tau}}\right\}\end{aligned}$$

At $y/a=0$ the short time solution becomes

$$\frac{u(0,\tau)}{u_0} = 1-2\left\{erfc\frac{1}{2\sqrt{\tau}} - erfc\frac{3}{2\sqrt{\tau}} + erfc\frac{5}{2\sqrt{\tau}} - \ldots\right\}$$

This series converges very quickly for small values of $\tau$.