UNIVERSITY OF WATERLOO

Department of Mechanical Engineering

ME 303 Advanced Engineering Mathematics

Problem Set 1

M.M. Yovanovich

---

1. Obtain the solution of the problem:

$$\frac{\partial^2 u}{\partial x^2} = xe^y; \quad u(0,y) = y^2, \quad u(1,y) = \sin y$$

   Answer:

$$u(x,y) = \frac{1}{6}x^3 e^y + x(\sin y - y^2 - \frac{1}{6}e^y) + y^2$$

2. Verify by substitution that

$$i) \quad u_1 = C_1 e^{-\lambda^2 \alpha t} \sin(\lambda x)$$

$$ii) \quad u_2 = C_2 e^{-\lambda^2 \alpha t} \cos(\lambda x)$$

   and the linear superposition of u_1 and u_2, i.e.,

$$iii) \quad u_3 = u_1 + u_2$$

   and

$$iv) \quad u(x,t) = C\, erfc\left(x/2\sqrt{\alpha t}\right)$$

   where

$$erfc\left(x/2\sqrt{\alpha t}\right) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{\alpha t}}^{\infty} e^{-\beta^2}\, d\beta$$

   are solutions of the one-dimensional diffusion equation

$$u_{xx} = \frac{1}{\alpha} u_t, \qquad t > 0$$

   The parameters C, C_1, C_2, α and λ^2 are constants.

3. Partial differential equations can be described and classified by their order (first, second, etc); their coefficients (constant or variable); and whether they are linear or non-linear; homogeneous or nonhomogeneous; parabolic, hyperbolic or elliptic.

   Describe and classify the following equations:

$$
\begin{aligned}
&(a) \qquad u_{xx} - nu_x - m^2 u = \frac{1}{\alpha} u_t \\
&(b) \qquad u_{xx} + u_{yy} + \lambda^2 u = 0 \\
&(c) \qquad c^2 u_{xx} - u_{tt} - hu_t = 0
\end{aligned}
$$

4. Verify that $u = C/r$ is a solution of the three-dimensional Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$

   where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, is the distance from the fixed point (or source point) (x_0, y_0, z_0) to the variable point (or field point) (x, y, z), and C is an arbitrary constant.

5. Verify by $u = C \ln r$ is a solution of the two-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0$$

   where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, is the distance from the fixed point (or source point) (x_0, y_0) to the variable point (or field point) (x, y), and C is an arbitrary constant.

6. Nondimensionalize the following partial differential equation (PDE), initial condition (IC) and boundary conditions (BCs) defined on the finite interval $[0, L]$:

$$\textbf{PDE} \qquad u_{xx} = \frac{1}{\alpha} u_t$$

$$\textbf{IC} \qquad u(x, 0) = u_0$$

$$
\begin{aligned}
&\textbf{BC1} \qquad u_x(0, t) = 0 \\
&\textbf{BC2} \qquad u(L, t) = u_1
\end{aligned}
$$

If the units of u, x and t are K, m and s, respectively, determine the units of the constant α. Let the dimensionless temperature be $\phi = (u - u_0)/(u_1 - u_0)$, the dimensionless position $\eta = x/L$, and the dimensionless time $\tau = \alpha t/L^2$.

7. Use Separation of Variables Method (SVM) on the given partial differential equations (PDEs) to obtained the three sets of independent ordinary differential equations (ODEs). *Do not attempt to solve the ODEs.* The independent spatial and temporal functions are denoted as $X(x), Y(y), Z(z), R(r)$ and $T(t)$. The separation constant is defined to be λ^2.

   - Two-dimensional Laplace equation in Cartesian coordinates.

$$u_{xx} + u_{yy} = 0$$

Set $u(x, y) = X(x)Y(y)$ and get

$$(i)\ X'' + \lambda^2 X = 0, \quad \text{and} \quad Y'' - \lambda^2 Y = 0$$

$$(ii)\ X'' - \lambda^2 X = 0, \quad \text{and} \quad Y'' + \lambda^2 Y = 0$$

$$(iii)\ X'' = 0, \quad \text{and} \quad Y'' = 0$$

   - Two-dimensional Laplace equation in cylindrical coordinates.

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0$$

Set $u(r, z) = R(r)Z(z)$ and get

$$(i)\ R'' + \frac{1}{r}R' + \lambda^2 R = 0, \quad \text{and} \quad Z'' - \lambda^2 Z = 0$$

$$(ii)\ R'' + \frac{1}{r}R' - \lambda^2 R = 0, \quad \text{and} \quad Z'' + \lambda^2 Z = 0$$

$$(iii)\ R'' + \frac{1}{r}R' = 0, \quad \text{and} \quad Z'' = 0$$

   - Two-dimensional Laplace equation in cylindrical coordinates.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi} = 0$$

Set $u(r, z) = R(r)\Phi(\phi)$ and get

$$(i)\ r^2 R'' + rR' + \lambda^2 R = 0, \quad \text{and} \quad \Phi'' - \lambda^2 \Phi = 0$$

$$(ii)\ r^2R'' + rR' - \lambda^2 R = 0, \quad \text{and} \qquad \Phi'' + \lambda^2\Phi = 0$$

$$(iii)\ r^2R'' + rR' = 0, \quad \text{and} \qquad \Phi'' = 0$$

- One-dimensional diffusion equation in Cartesian coordinates.

$$u_{xx} = \frac{1}{\alpha}u_t$$

Set $u(x,t) = X(x)T(t)$ and get

$$(i)\ X'' + \lambda^2 X = 0, \quad \text{and} \quad T' + \lambda^2\alpha T = 0$$

$$(ii)\ X'' - \lambda^2 X = 0, \quad \text{and} \quad T' - \lambda^2\alpha T = 0$$

$$(iii)\ X'' = 0, \quad \text{and} \quad T' = 0$$

- One-dimensional diffusion equation in cylindrical coordinates.

$$u_{rr} + \frac{1}{r}u_r = \frac{1}{\alpha}u_t$$

Set $u(r,t) = R(r)T(t)$ and get

$$(i)\ R'' + \frac{1}{r}R' + \lambda^2 R = 0, \quad \text{and} \qquad T' + \lambda^2\alpha T = 0$$

$$(ii)\ R'' + \frac{1}{r}R' - \lambda^2 R = 0, \quad \text{and} \qquad T' - \lambda^2\alpha T = 0$$

$$(iii)\ R'' + \frac{1}{r}R' = 0, \quad \text{and} \qquad T' = 0$$

- One-dimensional wave equation in Cartesian coordinates.

$$u_{xx} = \frac{1}{c^2}u_{tt}$$

Set $u(x,t) = X(x)T(t)$ and get

$$(i)\ X'' + \lambda^2 X = 0, \quad \text{and} \quad T'' + \lambda^2 c^2 T = 0$$

$$(ii)\ X'' - \lambda^2 X = 0, \quad \text{and} \quad T'' - \lambda^2 c^2 T = 0$$

$$(iii)\ X'' = 0, \quad \text{and} \quad T'' = 0$$

- One-dimensional wave equation in cylindrical coordinates.

$$u_{rr} + \frac{1}{r}u_r = \frac{1}{c^2}u_{tt}$$

Set $u(r,t) = R(r)T(t)$ and get

$$(i)\ R'' + \frac{1}{r}R' + \lambda^2 R = 0, \quad \text{and} \qquad T'' + \lambda^2 c^2 T = 0$$

$$(ii)\ R'' + \frac{1}{r}R' - \lambda^2 R = 0, \quad \text{and} \qquad T'' - \lambda^2 c^2 T = 0$$

$$(iii)\ R'' + \frac{1}{r}R' = 0, \quad \text{and} \qquad T'' = 0$$