Tutorial #6

Section 4.5

#27) $y''' - 6y'' - y' + 6y = 0$

Sub in
$y = e^{rx}$
$y' = re^{rx}$
$y'' = r^2e^{rx}$
$y''' = r^3e^{rx}$

$e^{rx}(r^3 - 6r^2 - r + 6) = 0$

$\therefore\ r^3 - 6r^2 - r + 6 = 0$ ⟸ need to solve a cubic.

To solve a cubic, an easy method is guess a value for the unknown, say 1, −1, 2, −2, etc, and sub it in. If LS = RS, then that is a correct root. Newton's Method can also be used.

Try $r=1$: $1^3 - 6(1)^2 - 1 + 6 = 0$ $\therefore\ r=1$ is a root.

Now, we can use this result to reduce the cubic to a quadratic.

$$
\begin{array}{r}
r^2 - 5r - 6 \\
(r-1)\ \overline{)\ r^3 - 6r^2 - r + 6} \\
\underline{r^3 - r^2 } \\
-5r^2 - r \\
\underline{-5r^2 + 5r } \\
-6r + 6 \\
\underline{-6r + 6} \\
0
\end{array}
$$

$\therefore\ (r-1)(r^2 - 5r - 6) = r^3 - 6r^2 - r + 6$

Now solve the quadratic.

$$r^2 - 5r - 6 = (r-6)(r+1)$$

Therefore, The roots are: $1, -1, 6$

The solutions are: Ae^x, Be^{-x}, Ce^{6x} (A, B, C are constants)

The general solution is a linear combination of these solutions:

$$\boxed{y = Ae^x + Be^{-x} + Ce^{6x}}$$

Section 4.8

#17) $y''(t) - 3y'(t) + 2y(t) = e^t \sin t$ (solved using the method of undetermined coefficients)

Step 1: Solve the complimentary equation

$$y'' - 3y' + 2y = 0$$
$$r^2 - 3r + 2 = 0$$
$$(r-2)(r-1) = 0$$
$$\therefore r = 2, 1$$
$$\therefore y_c = Ae^t + Be^{2t}$$

Step 2: Examine the RS of the equation and determine which type it fits on page 208.

RHS: $e^t \sin t$. $\Leftarrow$ Type 6.

$$\therefore y_p = Ce^t \cos t + De^t \sin t$$

Step 3: Sub y_p into original equation to try and solve for constants

$$y_p = Ce^t \cos t + De^t \sin t$$
$$y_p' = Ce^t \cos t - Ce^t \sin t + De^t \sin t + De^t \cos t$$
$$= e^t \cos t (C+D) + e^t \sin t (D-C)$$
$$y_p'' = e^t \cos t (C+D) - e^t \sin t (C+D) + e^t \sin t (D-C) + e^t \cos t (D-C)$$
$$= e^t \cos t (2D) + e^t \sin t (-2C)$$

$$\Rightarrow e^t \cos t (2D) + e^t \sin t (-2C) - 3e^t \cos t (C+D) - 3e^t \sin t (D-C) + 2Ce^t \cos t + 2De^t \sin t = e^t \sin t$$

$$\Rightarrow e^t \cos t (2D - 3C - 3D + 2C) + e^t \sin t (-2C - 3D + 3C + 2D) = e^t \sin t$$

$$\Rightarrow e^t \cos t (-D - C) + e^t \sin t (C - D) = e^t \sin t$$

$$\left.\begin{aligned} &\therefore -D - C = 0 \Rightarrow C = -D \quad ① \\ &\therefore C - D = 1 \Rightarrow C = 1 + D \quad ② \end{aligned}\right\} -D = 1 + D \left.\right\} \begin{aligned} D &= -\tfrac{1}{2} \\ C &= \tfrac{1}{2} \end{aligned}$$

$\therefore \quad y_p = \frac{1}{2}e^t \cos t - \frac{1}{2}e^t \sin t$

$\therefore \quad y_p = \frac{1}{2}e^t (\cos t - \sin t)$

Step 4: Combine Y_c and Y_p to form a general solution

$$\boxed{Y = Ae^t + Be^{2t} + \frac{1}{2}e^t (\cos t - \sin t)}$$ (ANS)

Section 4.8

#17) $y''(t) - 3y'(t) + 2y(t) = e^t \sin t$ (Solved using Variation of Parameters)

Step 1: Solve the complimentary equation $y'' - 3y' + 2y = 0$

$$r^2 - 3r + 2 = 0$$
$$(r-2)(r-1) = 0$$
$$\therefore r = 2, 1$$
$$\therefore y_c = Ae^t + Be^{2t} \Leftarrow \therefore y_1 = e^t \qquad y_2 = e^{2t}$$
$$\therefore y_1' = e^t \qquad y_2' = 2e^{2t}$$

Step 2: Use Y_c solutions to find particular solutions

$$y_p = u_1 y_1 + u_2 y_2$$

where. $u_1 = \int \frac{-y_2 f(t)}{W(y_1, y_2)} dt \qquad u_2 = \int \frac{y_1 f(t)}{W(y_1, y_2)} dt$

Solve for $W(y_1, y_2)$:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}$$

$$W = 2e^t e^{2t} - e^t e^{2t}$$
$$W = 2e^{3t} - e^{3t} = e^{3t}$$

Note that $f(t) = e^t \sin t$

$\therefore u_1 = \int \frac{-e^{2t}(e^t \sin t)}{e^{3t}} dt \qquad u_2 = \int \frac{e^t(e^t \sin t)}{e^{3t}} dt$

$u_1 = \int -\sin t \, dt \qquad u_2 = \int e^{-t} \sin t \, dt$

$u_1 = \cos t$ We have to use the u, v rule. See over →

$u_2 = \int e^{-t} \sin t \, dt$ let $u = e^{-t}$ $v = -\cos t$
$du = -e^{-t}$ $dv = \sin t$

$\therefore u_2 = uv - \int v\,du = -e^{-t}\cos t - \int e^{-t} \cos t \, dt$ let $u = e^{-t}$ $v = \sin t$
$du = -e^{-t}$ $dv = \cos t$

$\therefore u_2 = -e^{-t}\cos t - \left(e^{-t}\sin t + \int e^{-t} \sin t \, dt\right)$

$\therefore u_2 = -e^{-t}\cos t - e^{-t}\sin t - \int e^{-t} \sin t \, dt$

but $u_2 = \int e^{-t} \sin t \, dt$

$\therefore \int e^{-t} \sin t \, dt = -e^{-t}\cos t - e^{-t}\sin t - \int e^{-t} \sin t \, dt$

$2 \int e^{-t} \sin t \, dt = -e^{-t}\cos t - e^{-t}\sin t$

$\therefore \int e^{-t} \sin t \, dt = \frac{-e^{-t}}{2}(\cos t + \sin t) = u_2$

<u>sub back into yp</u>

$y_p = u_1 y_1 + u_2 y_2$
$y_p = e^t \cos t + e^{2t}\left(\frac{-e^{-t}}{2}\right)(\cos t + \sin t)$
$y_p = e^t \cos t - \frac{1}{2} e^t (\cos t) - \frac{1}{2} e^t (\sin t)$
$y_p = \frac{1}{2} e^t \cos t - \frac{1}{2} e^t \sin t$
$y_p = \frac{1}{2} e^t (\cos t - \sin t)$

<u>Step 3: Combine y_c and y_p to form general solution y</u>

$y = y_c + y_p$
$\boxed{y = Ae^t + Be^{2t} + \frac{1}{2} e^t (\cos t - \sin t)}$ (ANS)

Section 4.9

#21) $x^2 z'' - xz' + z = x\left(1 + \frac{3}{\ln x}\right)$

NOTE: The presence of the x terms beside the z terms indicates that this is of Cauchy-Euler form. ∴ substitute $x = e^t$ to whole equation right from the start.

$$e^{2t} z'' - e^t z' + z = e^t\left(1 + \frac{3}{t}\right)$$

Step 1: Solve the homogeneous Equation

$$e^{2t} z'' - e^t z' + z = 0$$

Since we already realized that this is of Cauchy-Euler form, and we subbed in $x = e^t$, we can simplify the expression according to the Cauchy-Euler method:

$$\left.\begin{aligned} x &= e^t \\ xz' &= \frac{dz}{dt} \\ x^2 z'' &= \frac{d^2 z}{dt^2} - \frac{dz}{dt} \end{aligned}\right\}$$ See pages 187-188 for this derivation.

$$\therefore \quad \frac{d^2 z}{dt^2} - \frac{dz}{dt} - \frac{dz}{dt} + z = 0$$

$$z'' - 2z' + z = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$\therefore r = 1$$ ⟸ two repeated roots !!

$$\therefore z_c = Ae^t + Bte^t$$

The complementary z solutions are: $z_1 = e^t$ $\quad z_2 = te^t$

$z_1' = e^t \quad z_2' = e^t + te^t$

Step 2 - Use z_c solutions to find Particular solution z_p

$$z_p = u_1 z_1 + u_2 z_2$$

where $u_1 = \int \frac{-z_2 f(t)}{W(z_1, z_2)} dt \quad u_2 = \int \frac{z_1 f(t)}{W(z_1, z_2)} dt$

Solve for $W(z_1, z_2)$

$$W = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}$$

Solve for u_1 and u_2

$$u_1 = -\int \frac{(te^t)(e^t)(1+\frac{3}{t})}{e^{2t}} dt \qquad u_2 = \int \frac{e^t(e^t)(1+\frac{3}{t})}{e^{2t}} dt$$

$$u_1 = -\int t+3 \, dt \qquad u_2 = \int 1 + \frac{3}{t} \, dt$$

$$u_1 = -\frac{t^2}{2} + 3t \qquad u_2 = t + 3\ln t$$

$$z_p = \left(-\frac{t^2}{2} + 3t\right)e^t + (t + 3\ln t)te^t$$
$$z_p = -\frac{t^2}{2}e^t + 3te^t + t^2e^t + 3te^t \ln t$$
$$z_p = \frac{1}{2}t^2e^t + 3te^t + 3te^t \ln t$$

Step 3 - Combine z_c and z_p to form z

$$z = Ae^t + Bte^t + \frac{1}{2}t^2e^t + 3te^t + 3te^t \ln t$$
$$z = Ae^t + Cte^t + \frac{1}{2}t^2e^t + 3te^t \ln t \qquad (C = B+3)$$

Sub back in for x

$$\left.\begin{aligned} x &= e^t \\ \ln x &= t \\ \ln[\ln x] &= \ln t \end{aligned}\right\} \boxed{z = Ax + Cx\ln x + \frac{1}{2}(\ln x)^2 x + 3x(\ln x)\ln[\ln(x)]}$$