Tutorial #5

Section 4.3

#7) $x^2 y'' - 2y = 0$

solutions: $y_1 = x^2$, $y_2 = x^{-1}$

boundary conditions: $y(1) = -2$, $y'(1) = -7$

a) check linear independence

Method #1: Ensure Wronskian $\neq 0$

$$\text{Wronskian} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Determine y_1 y_1' y_2 and y_2': $y_1 = x^2$ $y_2 = x^{-1}$
$y_1' = 2x$ $y_2' = -x^{-2}$

Compute Wronskian: $(x^2)(-x^{-2}) - (x^{-1})(2x) = -1 - 2 = -3$

$\boxed{\text{Since Wronskian} \neq 0\text{, the two solutions } y_1 \text{ and } y_2 \text{ are linearly independent}}$

Method #2: Ensure $y_1 \neq c\, y_2$ for all x

Sub in $x=1$ $(1)^2 = c(1)^{-1}$ $\therefore c = 1$

Sub in $x=2$ $(2)^2 = c(2)^{-1}$ $\therefore c = 8$

$\boxed{\text{Since there is not only one value for } c \text{ for all } x\text{, the solutions are linearly independent}}$

b) Find a general solution

First check given solutions: $y_1 = x^2$ $\quad$ $y_2 = x^{-1}$

$y_1' = 2x$ $\quad$ $y_2' = -x^{-2}$

$y_1'' = 2$ $\quad$ $y_2'' = 2x^{-3}$

check: $x^2(2) - 2(x^2) = 0$ $\therefore$ Okay $\quad$ check: $x^2(2x^{-3}) - 2(x^{-1}) = 0$ $\therefore$ okay

If the two solutions are indeed solutions, and they are linearly independent, then we can form a linear combination of the two solutions:

$$y_3 = Ay_1 + By_2 \qquad (A, B \text{ constants})$$

$$y_3 = Ax^2 + Bx^{-1}$$

$$y_3' = 2Ax - Bx^{-2}$$

c) Find a specific solution

$y(1) = -2$ $\quad$ $-2 = A(1)^2 + B(1)^{-1} = A + B$ $\quad$ $\therefore A = -B - 2$ ①

$y'(1) = -7$ $\quad$ $-7 = 2A(1) - B(1)^{-2} = 2A - B$ ②

① → ② $\quad$ $-7 = 2(-B-2) - B$

$-7 = -2B - 4 - B$

$-3 = -3B$

$B = 1$ $\quad$ $\therefore A = -3$

$\boxed{\therefore \text{ The specific solution is: } \; y = -3x^2 + x^{-1}}$

Section 4.5

#(5) $y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y'(0) = -3$

Sub in $y = e^{mx}$

$$y' = me^{mx}$$
$$y'' = m^2 e^{mx}$$

$\therefore\ e^{mx}(m^2 + 2m + 1) = 0$

$\therefore\ m^2 + 2m + 1 = 0$ (since e^{mx} can never be 0)

Solve the quadratic: $(m+1)^2 = 0$

$\therefore m = -1$ (two repeated roots !!)

If there were two different real roots, we would have the solution

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

But, with one repeated real root, we instead have

$$y = Ae^{mx} + Bxe^{mx}$$
$$y = Ae^{-x} + Bxe^{-x}$$
$$y' = -Ae^{-x} - Bxe^{-x} + Be^{-x}$$

$y(0) = 1 \qquad 1 = Ae^0 + B(0)e^0 = A \qquad \boxed{\therefore A = 1}$

$y'(0) = -3 \qquad -3 = -Ae^0 - B(0)e^0 + Be^0 = -A + B \qquad \boxed{\therefore B = -2}$

$\boxed{\therefore \text{The specific solution is } y = e^{-x} - 2xe^{-x}}$

Section 4.6

#21) $y'' + 2y' + 2y = 0$ $\quad y(0)=2 \quad y'(0)=1$

rewrite as: $m^2+2m+2=0$

Solve for m: $m = -2 \pm \sqrt{2^2-4(2)}\ /2$

$m = -2 \pm \sqrt{-4}\ /2$

$m = -2 \pm 2i\ /2$

$m = -1 \pm i$ (two complex roots)

$y = Ae^{REAL\,x}\cos(IMAG\,x) + Be^{REAL\,x}\sin(IMAG\,x)$ ⇐ general form

$y = Ae^{-x}\cos(x) + Be^{-x}\sin(x)$

$y' = -Ae^{-x}\cos(x) - Ae^{-x}\sin(x) - Be^{-x}\sin(x) + Be^{-x}\cos(x)$

$= e^{-x}\cos(x)[B-A] + e^{-x}\sin(x)[-A-B]$

$y(0)=2 \quad 2 = Ae^0\cos(0) + Be^0\cancel{\sin(0)} = A \quad \boxed{\therefore A=2}$

$y'(0)=1 \quad 1 = e^0\cos(0)[B-A] + e^0\cancel{\sin(0)}[-A-B] = B-A \quad \boxed{\therefore B=3}$

$$\boxed{\therefore y = 2e^{-x}\cos(x) + 3e^{-x}\sin(x)}$$

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