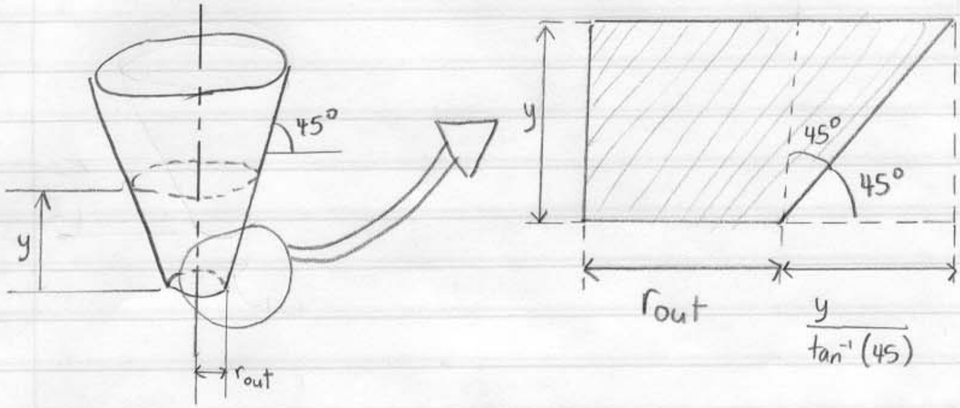
Tutorial #4

Q1) First, find an expression for the Volume in the conical cylinder at height y.



$$\therefore\ r(y) = r_{out} + \frac{y}{\tan^{-1}(45)}$$

$$r(y) = r_{out} + y$$

$$\forall = \int_0^y \pi r(y)^2\, dy$$

$$\forall = \pi \int_0^y (r_{out} + y)^2\, d_y$$

$$\forall = \pi \left[\tfrac{1}{3} (r_o + y)^3 \right]_0^y$$

$$\boxed{\forall = \frac{\pi}{3} \left[(r_o + y)^3 - r_o^{\,3} \right]}$$

Use Conservation of mass to relate flow into and out of the cylinder

$$\frac{\partial m}{\partial t} = \dot{m}_{in} - \dot{m}_{out}$$

$$\frac{\partial}{\partial t}(\rho \forall) = \rho Q_{in} - \rho Q_{out}$$

$$\frac{\partial}{\partial t}\left(\frac{\pi}{3} \left[(r_o + y)^3 - r_o^3 \right] \right) = Q_{in} - c\pi r_o^2 \sqrt{2g}\sqrt{y} \qquad \left(\text{Let } A = c\pi r_o^2 \sqrt{2g} \right)$$

$$\frac{\pi}{3} \left[3 (r_o + y)^2 \frac{dy}{dt} \right] = Q_{in} - A\sqrt{y}$$

(a) $\Longrightarrow$ $\boxed{\pi (r_o + y)^2 \frac{dy}{dt} = Q_{in} - A\sqrt{y}}$

b) Use definite integrals to take into account the initial condition $y(0)=0$ and the boundary condition $y(t)=L$.

$$\int_0^L \frac{\pi (r_0+y)^2}{Q_{in}-A\sqrt{y}}\,dy = \int_0^t dt$$

(b) $\Longrightarrow$ $\boxed{\therefore\ t = \pi \int_0^L \frac{(r_0+y)^2}{Q_{in}-A\sqrt{y}}\,dy}$

c) If the oil level is steady, then $\frac{\partial m}{\partial t}=0$. Therefore...

$$0 = \dot{m}_{in} - \dot{m}_{out}$$

$$\rho Q_{in} = \rho Q_{out}$$

$$Q_{in} = c A_o \sqrt{2gy}$$

Sub in values, including $y=0.30$

$$Q_{in} = (0.6)\pi(0.05)^2\sqrt{2(9.81)(0.30)}$$

$$Q_{in} = 0.0114\ (m^3/s) = 0.0114\ (L/s)$$

$$\boxed{\therefore\ Q_{in} = 0.0114\,(L/s)\ \text{when}\ y = 30\ cm}$$

d) If the inflow suddenly stops, then $Q_{in}=0$. Evaluate the integral of part (b), but use the limits of y going from $30\ cm \rightarrow 0\ cm$.

$$t = \pi \int_{0.30}^{0} \frac{(0.05+y)^2}{-A\sqrt{y}}\,dy$$

$$t = \frac{-\pi}{A}\int_{0.30}^{0}\left(0.0025\,y^{-\frac{1}{2}} + 0.1\,y^{1/2} + y^{3/2}\right)dy$$

$$t = \frac{-\cancel{\pi}}{(0.6)\cancel{\pi}(0.05)^2\sqrt{2(9.81)}}\left[0.005y^{1/2} + \frac{0.2}{3}y^{3/2} + \frac{2}{5}y^{5/2}\right]_{0.30}^{0}$$

$$t = -150.51\left[-0.03341\right]$$

$t =$ 5.03 seconds

(d) $\Longrightarrow$ $\boxed{\text{∴ It would take 5.03 seconds to drain the container}}$

Q2) $$\frac{dP}{dz} = -\frac{Pg}{R(T_0 - \gamma z)} \qquad P(z=0) = P_0$$

a) Find S.I. units of R and γ.

$(T_0 - \gamma z)$ ← for this minus operation to take place, both terms must be in the same units.

(a) $\Longrightarrow$ $\boxed{\therefore\ \gamma_0 \text{ must have units of } \left[\frac{k}{m}\right]}$

Perform a unit balance on the RHS of the expression.

$$\frac{\left[\frac{N}{m^2}\right]\left[\frac{m}{s^2}\right]}{[R][k]} = \frac{\left[\frac{N}{m^2}\right]}{[m]}$$

(a) $\Longrightarrow$ $\boxed{\therefore\ [R] = \left[\frac{m^2}{s^2 k}\right]}$

b) $\rho^* = \dfrac{\rho - \rho_r}{\Delta_\rho} \Rightarrow d\rho^* = \dfrac{d\rho}{\Delta_\rho} \Rightarrow d\rho = \Delta_\rho d\rho^*$

$$z^* = \frac{z - z_r}{\Delta_z} \Rightarrow dz^* = \frac{dz}{\Delta_z} \Rightarrow dz = \Delta_z dz^*$$

$$\therefore \frac{d\rho}{dz} = \frac{\Delta_\rho}{\Delta_z}\frac{d\rho^*}{dz^*} = -\frac{(\Delta_\rho \rho^* + \rho_r)\, g}{R\,(T_0 - \gamma(\Delta_z z^* + z_r)}$$

$$\frac{d\rho^*}{dz^*} = \frac{-\left(\rho^* + \frac{\rho_r}{\Delta_\rho}\right) g}{\frac{RT_0}{\Delta_z} - R\gamma z^* - \frac{R\gamma z_r}{\Delta z}}$$

let $\rho_r = z_r = 0$ to get rid of the $\frac{\rho_r}{\Delta_\rho}$ and $\frac{R\gamma z_r}{\Delta z}$ terms.
Also, since $\rho = \rho_0$ when $z=0$, we should normalize ρ^* to make $\rho^* = 1$ when $z^* = 0$.

$$\therefore \rho^* = \frac{\rho - \rho_r}{\Delta_\rho} \quad (\text{sub in } \rho_r = 0,\ \rho^* = 1,\ \rho = \rho_0)$$

$$1 = \frac{\rho_0}{\Delta_\rho} \Rightarrow \boxed{\Delta_\rho = \rho_0}$$

$$\therefore \frac{d\rho^*}{dz^*} = \frac{-\rho^* g}{\frac{RT_0}{\Delta_z} - R\gamma z^*} = \frac{-\rho^* g}{R\gamma\left(\frac{T_0}{\gamma\Delta_z} - z^*\right)}$$

Now let $\frac{T_0}{\gamma \Delta_z} = 1$ to simplify things. $\boxed{\therefore\ \Delta_z = \frac{T_0}{\gamma}}$

$$\text{(b)} \Rightarrow \boxed{\therefore \frac{d\rho^*}{dz^*} = \frac{-\rho^* g}{R\gamma(1 - z^*)}}$$

c) Solve the ODE

$$\int \frac{dp^*}{p^*} = \frac{-g}{R\gamma} \int \frac{dz^*}{1-z^*}$$

$$\ln p^* = \frac{g}{R\gamma} \ln (1-z^*) + C$$

$$p^* = e^C (1-z^*)^{g/R\gamma} \qquad (\text{Let } D = e^C)$$

$$p^* = D (1-z^*)^{g/R\gamma}$$

when $z^* = 0$, $p^* = 1$ so...

$$\boxed{1 = D}$$

$$(c) \Rightarrow \boxed{\therefore p^* = (1-z^*)^{g/R\gamma}}$$

Sub back in for P and z

$$\frac{P}{P_0} = \left(1 - \frac{z\gamma}{T_0}\right)^{g/R\gamma}$$

$$(c) \Rightarrow \boxed{P = P_0 \left(1 - \frac{z\gamma}{T_0}\right)^{g/R\gamma}}$$