Tutorial #3

Section 2.5

#11) $(y^2+2xy)\,dx - x^2\,dy = 0$

$\Rightarrow$ can't separate

$\Rightarrow$ can't put in form $\frac{dy}{dx} + P(x)\,y = Q(x)$ due to y^2 term

$\Rightarrow$ check exactness:

$$\left.\begin{array}{ll} M = y^2 + 2xy & \frac{\partial M}{\partial y} = 2y + 2x \\ N = -x^2 & \frac{\partial N}{\partial x} = -2x \end{array}\right\} \therefore \text{ not exact}$$

$\Rightarrow$ examine $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2y+2x+2x}{-x^2} = \dfrac{2y+4x}{-x^2}$ } not only a function of x

$\Rightarrow$ examine $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{-2x-2y-2x}{y^2+2xy} = \dfrac{-4x-2y}{y(y+2x)} = \dfrac{-2\cancel{(y+2x)}}{y\cancel{(y+2x)}}$ } This is a fcn of y only so use special integrating facto

<u>Special Integrating factor</u>

$$u = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right) = \exp\int \frac{-2}{y}\,dy = e^{-2\ln y} = y^{-2}$$

<u>Multiply both sides of the original equation by y^{-2}</u>

$$(1 + 2xy^{-1})\,dx - (x^2y^{-2})\,dy = 0$$

$\Rightarrow$ check exactness again:

$$\left.\begin{array}{ll} M = 1 + 2xy^{-1} & \frac{\partial M}{\partial y} = -2xy^{-2} \\ N = -x^2y^{-2} & \frac{\partial N}{\partial x} = -2xy^{-2} \end{array}\right\} \text{Now the equation is exact.}$$

$\Rightarrow$ compute F:

$$F = \int M\,dx + g(y) \qquad\qquad F = \int N\,dy + h(x)$$

$$F = \int 1 + 2xy^{-1}\,dx + g(y) \qquad\qquad F = \int -x^2y^{-2}\,dy + h(x)$$

$$F = x + x^2y^{-1} + g(y) \qquad\qquad F = x^2y^{-1} + h(x)$$

$\Rightarrow$ Combining the expressions for F: $\quad F = x + x^2y^{-1} = C$

$$\boxed{\therefore\ y = \frac{x^2}{C-x}}$$

Check solution: $y = \frac{x^2}{c-x}$

$$\frac{dy}{dx} = \frac{2x}{c-x} + \frac{x^2}{(c-x)^2}$$

rearrange original function: $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$

LS	RS
dy/dx	$\frac{y^2+2xy}{x^2}$
$\frac{2x}{c-x} + \frac{x^2}{(c-x)^2}$	$\frac{\left(\frac{x^2}{c-x}\right)^2 + 2x\left(\frac{x^2}{c-x}\right)}{x^2}$
$\frac{x^2}{(c-x)^2} + \frac{2x}{c-x}$	$\left[\frac{x^4}{(c-x)^2} + \frac{2x^3}{c-x}\right]\frac{1}{x^2}$
$\frac{x^2}{(c-x)^2} + \frac{2x}{c-x}$	$\frac{x^2}{(c-x)^2} + \frac{2x}{c-x}$

$\Leftarrow$ LS = RS so the Solution is correct.

Section 2.6

#11) $(y^2 - xy)\,dx + x^2\,dy = 0$

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2 \quad \Big\}\ \text{a fcn of } \left(\frac{y}{x}\right) \text{ so sub in } v = \frac{y}{x}$$

if $v = \frac{y}{x}$, then $y = xv$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Sub into original:

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} - v^2$$

$$\frac{dv}{dx} = \frac{-v^2}{x}$$

$$\int -\frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{v} = \ln x + C$$

sub $v = \frac{y}{x}$

$$\frac{x}{y} = \ln x + C$$

$$\boxed{y = \frac{x}{\ln x + C}}$$

Check solution

$$y = \frac{x}{\ln x + c}$$

$$\frac{dy}{dx} = \frac{1}{\ln x + c} \quad \frac{-x\left(\frac{1}{x}\right)}{(\ln x + c)^2} = \frac{1}{\ln x + c} - \frac{1}{(\ln x + c)^2}$$

LS

$$\Rightarrow \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{\ln x + c} - \frac{1}{(\ln x + c)^2}$$

RS

$$\Rightarrow \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

$$\Rightarrow \frac{1}{\ln x + c} - \left(\frac{1}{\ln x + c}\right)^2$$

$\therefore$ The solution is correct.

Section 2.6

#19) $\frac{dy}{dx} = (x-y+5)^2$

Sub in $z = x-y+5$ $\quad \therefore \frac{dz}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dz}{dx}$

rewrite: $1 - \frac{dz}{dx} = z^2$

$$\frac{dz}{dx} = 1 - z^2 = (1-z)(1+z)$$

$$\frac{dz}{(1-z)(1+z)} = dx$$

$$\frac{1}{2}\int \frac{1}{1-z} + \frac{1}{1+z} = \int dx \quad \text{[see Aside:]}$$

$$\frac{1}{2}\left(-\ln|1-z| + \ln|1+z|\right) = x + c$$

$$e^{-\ln|1-z|} e^{\ln|1+z|} = e^{2(x+c)}$$

$$|1-z|^{-1}|1+z| = e^{2x}e^{2c}$$

let $D = e^{2c}$

$$\frac{|1+z|}{|1-z|} = De^{2c}$$

Aside:

$$\frac{A}{(1-z)} + \frac{B}{(1+z)} = \frac{1}{(1-z)(1+z)}$$

$$\therefore A + Az + B - Bz = 1$$

$$\therefore Az - Bz = 0 \Rightarrow A = B$$

$$\therefore A + B = 1 \Rightarrow A = 1 - B$$

$$\therefore A = \tfrac{1}{2}, \quad B = \tfrac{1}{2}$$

$$\therefore \frac{1}{2(1-z)} + \frac{1}{2(1+z)} = \frac{1}{(1-z)(1+z)}$$

if $\frac{|1+z|}{|1-z|} = \frac{1+z}{1-z}$ then

$$\Rightarrow \frac{1+x-y+5}{1-x+y-5} = De^{2x}$$

$$\Rightarrow 6+x-y = De^{2x}(-x+y-4)$$

$$\Rightarrow y(-1-De^{2x}) = -(x+4)De^{2x} - (x+6)$$

$$\Rightarrow y(De^{2x}+1) = (x+4)De^{2x} + (x+6)$$

$$\Rightarrow y = \frac{(x+4)De^{2x} + (x+6)}{De^{2x}+1}$$

$$\Rightarrow y = \frac{x(De^{2x}+1) + 6 + 4De^{2x}}{De^{2x}+1}$$

$$\Rightarrow \boxed{y = x + \frac{6+4De^{2x}}{De^{2x}+1}}$$

if $\frac{|1+z|}{|1-z|} = \frac{-(1+z)}{(1-z)}$ then

$$\Rightarrow \frac{1+x-y+5}{-1+x-y+5} = De^{2x}$$

$$\Rightarrow 6+x-y = De^{2x}(x-y+4)$$

$$\Rightarrow y(-1+De^{2x}) = (x+4)De^{2x} - (6+x)$$

$$\Rightarrow y = \frac{x(De^{2x}-1) - 6 + 4De^{2x}}{(De^{2x}-1)}$$

$$\Rightarrow \boxed{y = x + \frac{-6+4De^{2x}}{De^{2x}-1}}$$

check:

if $y = x + \dfrac{6+4De^{2x}}{De^{2x}+1}$ Then

$$\frac{dy}{dx} = 1 + \frac{8De^{2x}}{De^{2x}+1} - \frac{(6+4De^{2x})(2De^{2x})}{(De^{2x}+1)^2}$$

$$\frac{dy}{dx} = \left[\frac{De^{2x}+1+8De^{2x}}{De^{2x}+1}\right]\left(\frac{De^{2x}+1}{De^{2x}+1}\right) - \frac{(6+4De^{2x})(2De^{2x})}{(De^{2x}+1)^2}$$

$$\frac{dy}{dx} = \frac{9(De^{2x})^2 + 9De^{2x} + De^{2x} + 1 - 12De^{2x} - 8(De^{2x})^2}{(De^{2x}+1)^2}$$

$$\frac{dy}{dx} = \frac{(De^{2x})^2 - 2(De^{2x}) + 1}{(De^{2x}+1)^2}$$

$$\frac{dy}{dx} = \frac{(De^{2x}-1)^2}{(De^{2x}+1)^2}$$

look at original equation

LS

$\Rightarrow \dfrac{dy}{dx}$

$\Rightarrow \dfrac{(De^{2x}-1)^2}{(De^{2x}+1)^2}$

RS

$\Rightarrow (x-y+5)^2$

$\Rightarrow \left(x - x \dfrac{-6-4De^{2x}}{De^{2x}+1} + 5\right)^2$

$\Rightarrow \dfrac{(-6-4De^{2x}+5De^{2x}+5)^2}{(De^{2x}+1)^2}$

$\Rightarrow \dfrac{(De^{2x}-1)^2}{(De^{2x}+1)^2}$

Since LS = RS, the equation is a solution.

NOTE: A similar check can be performed on the $y = x + \dfrac{-6+4De^{2x}}{De^{2x}-1}$ solution.