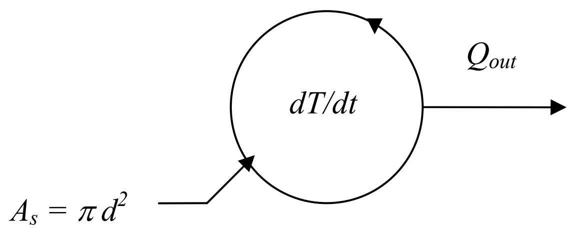
ME203 – Ordinary Differential Equations October 1, 2002
Mathcad Demonstration – Applications of Solutions for First Order ODEs

Example #1 – Transient Heat Transfer

Steel spheres 12 *mm* in diameter are annealed by heating to 1150*K* and then slowly cooled to 400*K* in an air environment for which $T_{air} = 325K$ and $h = 20 W/m^2 K$. Assuming the following properties for steel, calculate the time required for the cooling process:



$\rho = 7800\, kg/m^3$

$c_p = 600\, J/kg\, K$

Solution

1. Identify variables: dependent variable = temperature, *T* [*K*]
   independent variable = time, *t* [*s*]

2. Fundamental Laws:

Conservation of Energy

Change in Internal Energy = Energy Outflow

$$m\, c_p \frac{dT}{dt} = -Q_{out}$$

Newtons Law of Cooling

$$Q_{out} = h\, A_s \left(T - T_{air}\right)$$

where *h* is called the convective heat transfer coefficient (constant)

First order, linear separable ODE

$$m\, c_p \frac{dT}{dt} = -h\, A_s \left(T - T_{air}\right)$$

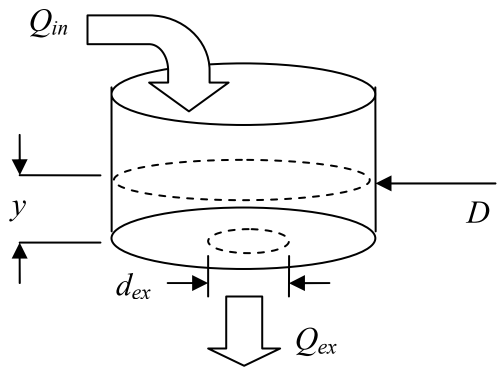
$$\frac{dT}{dt} = -\frac{h\, A_s}{m\, c_p}\left(T - T_{air}\right)$$

Initial condition

$$T(t = 0) = 1150K$$

Example #2 – Tank Draining Problem

Water enters a cylindrical tank of diameter $D = 1\,m$ at $Q_{in} = 0.0026\,m^3/s$. There is a small hole of diameter $d_{ex} = 5\,cm$ at the bottom of the tank. Find an equation for the water level in the tank *y(t)*, given initial condition $y(t = 0) = 0.5\,m$



Solution

1. Identify variables — dependent variable = height of fluid, y [m]
   independent variable = time, t [s]
2. Fundamental Laws:

Conservation of Mass

Change in Tank Volume = Inflow – Outflow

$$\rho \frac{dV}{dt} = \rho(Q_{in} - Q_{ex})$$

Tank volume $V = \frac{\pi}{4} D^2 \cdot y$, $\frac{dV}{dt} = \frac{\pi}{4} D^2 \cdot \frac{dy}{dt}$

Torricelli's Law

$$v_{ex} = C_d \sqrt{2\,g\,y}$$

where C_d is the discharge coefficient (constant)

Volume flowrate of outflow $Q_{ex} = A_{ex} \cdot v_{ex} = \frac{\pi\, d_{ex}^2}{4} \cdot C_d \sqrt{2\,g\,y}$

First order, non-linear ODE

$$\rho \frac{\pi}{4} D^2 \cdot \frac{dy}{dt} = \rho \left(Q_{in} - \frac{\pi\, d_{ex}^2}{4} \cdot C_d \sqrt{2\,g\,y} \right)$$

$$\frac{dy}{dt} = \frac{4 Q_{in}}{\pi D^2} - C_d \sqrt{2g}\, \frac{d_{ex}^2}{D^2}\, y^{\frac{1}{2}}$$

$$y(t = 0) = 0.5\,m$$

3. Solution of ODE

$$\text{Substitute: } \alpha = C_d \sqrt{2g}\, \frac{d_{ex}^2}{D^2} \quad , \quad \beta = \frac{4Q_{in}}{\pi D^2}$$

$$\text{ODE becomes: } \frac{dy}{dt} + \alpha\, y^{\frac{1}{2}} = \beta$$

First order, non-linear, separable

$$\frac{dy}{\beta - \alpha\, y^{\frac{1}{2}}} = dt$$

$$\text{Substitute: } y = u^2 \ , \ dy = 2u\, du$$

$$\frac{2u\, du}{\beta - \alpha\, u} = dt$$

Factor $-2/\alpha$ from ODE

$$-\frac{2}{\alpha}\left(\frac{u\, du}{u - \beta/\alpha}\right) = dt$$

$$-\frac{2}{\alpha}\left(\frac{(u - \beta/\alpha + \beta/\alpha)du}{u - \beta/\alpha}\right) = dt$$

$$-\frac{2}{\alpha}\left(\frac{u - \beta/\alpha}{u - \beta/\alpha} + \frac{\beta/\alpha}{u - \beta/\alpha}\right)du = dt$$

Integrate both sides:

$$u + \frac{\beta}{\alpha} \cdot \ln\left| u - \frac{\beta}{\alpha} \right| = -\frac{\alpha\, t}{2} + C_1$$

Multiply both sides by α/β and take exponential of both sides:

$$\left(u - \frac{\beta}{\alpha}\right) e^{\frac{\alpha u}{\beta}} = C_3\, e^{\frac{-\alpha^2 t}{2\beta}}$$

Substitute initial condition and back substitute in terms of y:

$$\frac{1 - \frac{\alpha}{\beta}\sqrt{y}}{1 - \frac{\alpha}{\beta}\sqrt{y_0}}\, e^{\frac{\alpha}{\beta}\left(\sqrt{y} - \sqrt{y_0}\right)} = e^{\frac{-\alpha^2}{2\beta} t}$$