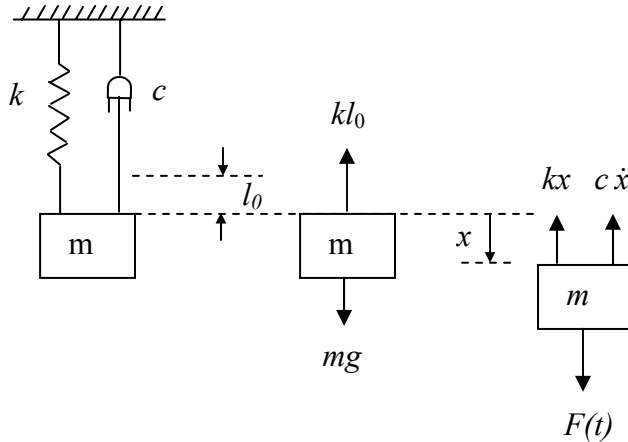


**ME203 Section 4.1 – Forced Vibration Response of Linear System Nov 4, 2002**

When a linear mechanical system is excited by an external force, its response will depend on the form of the excitation force  $F(t)$  and the amount of damping which is present. The free-body diagram is shown below. The equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{1}$$



The solution to equation (1) will have two parts. If  $F(t) = 0$ , we have the **homogeneous** differential equation whose solution corresponds physically to damped free vibration. This is also called the **transient** solution, since it will always eventually die out in time. (Recall from earlier class that the homogeneous solution is multiplied by an exponential function with negative exponent – this is the effect of damping.)

When  $F(t) \neq 0$ , we obtain a particular solution (the forced response) that is due to the excitation. This is also called the **steady-state** solution and persists as long as  $F(t)$  is present.

**Review of Transient Solution Procedure**

When  $F(t) = 0$ , we get the homogeneous equation:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2}$$

The traditional solution method (since  $m$ ,  $c$  and  $k$  are all constants) is to assume a solution of the form  $x = e^{rt}$ , which yields the characteristic equation:

$$mr^2 + cr + k = 0 \tag{3}$$

The roots of this equation are:

$$r_1, r_2 = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad (4)$$

and the general solution to the homogeneous equation is given by:

$$\begin{aligned} x(t) &= Ae^{r_1 t} + Be^{r_2 t} \\ &= e^{-(c/2m)t} \left( Ae^{\left(\sqrt{(c/2m)^2 - k/m}\right)t} + Be^{-\left(\sqrt{(c/2m)^2 - k/m}\right)t} \right) \end{aligned} \quad (5)$$

The first term  $e^{-(c/2m)t}$  is simply an exponentially decaying function of time. The behaviour of the terms in brackets depends on whether the discriminant is positive, zero or negative.

When  $c^2 > 4km$ , the exponents of the bracketed terms are real numbers and no oscillations are possible. This case is referred to as **overdamped**. An overdamped system rapidly returns to equilibrium when perturbed.

When  $c^2 < 4km$ , the exponents become imaginary numbers  $\pm i\sqrt{k/m - (c/2m)^2} t$ . The result can then be expressed as sines and cosines, viz.:

$$x(t) = e^{-(c/2m)t} \left( A \cos(\sqrt{k/m - (c/2m)^2} t) + B \sin(\sqrt{k/m - (c/2m)^2} t) \right) \quad (6)$$

**Critical Damping.** For the limiting case between the oscillatory and non-oscillatory motion, we define critical damping as the value of  $c$  which gives a zero discriminant:

$$\left(\frac{c_c}{2m}\right)^2 = \frac{k}{m} = \omega_n^2 \quad (7)$$

or

$$c_c = 2\sqrt{km} = 2m\omega_n \quad (8)$$

Here  $\omega_n = \sqrt{k/m}$  is the **natural frequency** of vibration of the **undamped** system. It is convenient to define the *damping ratio*  $\zeta$ ,

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} \quad (9)$$

so that

$$\frac{c}{2m} = \zeta \omega_n \quad (10)$$

With this relation, the roots of the characteristic equation (4) become:

$$r_1, r_2 = \omega_n (-\zeta \pm \sqrt{\zeta^2 - 1}) \quad (11)$$

The three cases of damping now depend on whether  $\zeta > 1$  (overdamped),  $\zeta < 1$  (underdamped) or  $\zeta = 1$  (critically damped). We are usually most interested in the underdamped case, with  $\zeta < 1$ . For this case the roots are complex:

$$r_1, r_2 = \omega_n (-\zeta \pm i\sqrt{1 - \zeta^2}) \quad (12)$$

and

$$x(t) = e^{-\zeta \omega_n t} \left( A \cos(\sqrt{1 - \zeta^2} \omega_n t) + B \sin(\sqrt{1 - \zeta^2} \omega_n t) \right) \quad (13)$$

This can also be written as,

$$x(t) = e^{-\zeta \omega_n t} X \cos(\sqrt{1 - \zeta^2} \omega_n t - \phi) \quad (14)$$

where

$$X = \sqrt{A^2 + B^2} \quad \text{and} \quad \phi = \text{Tan}^{-1} \left( \frac{B}{A} \right) \quad (15)$$

N.B. Note that the angle  $\phi$  in equation (14) is always expressed in radians!!

Equation (14) indicates that the **frequency of damped oscillation** is equal to

$$\mu = \omega_n \sqrt{1 - \zeta^2} \quad (16)$$

For small damping ( $\zeta \ll 1$ ), we can expand (16) in a binomial series and take only the lead term:

$$\mu \cong \omega_n \left( 1 - \frac{1}{2} \zeta^2 \right) = \omega_n \left( 1 - \frac{c^2}{8km} \right) \quad (17)$$

(In equation (17) we have used the binomial series  $(1 + \varepsilon)^n \cong 1 + n\varepsilon + \frac{n(n-1)}{2} \varepsilon^2 + \dots$ )

Thus we see that the frequency of oscillation of the damped system is always **less** than that of the undamped system. The general behaviour of the underdamped solution (equation 14) is plotted below in Figure 1.

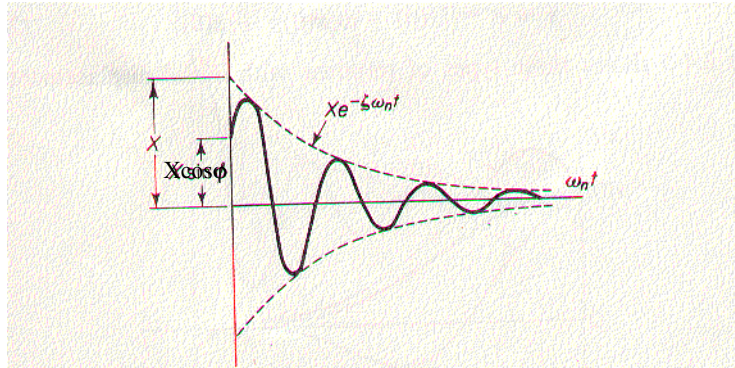


Figure 1. Damped oscillation behaviour for  $\zeta < 1$ .

### Forced Oscillations

The solution (13) to the homogeneous (unforced) equation of motion is sometimes called the **transient** solution, since it will always decay to zero after sufficient time has passed. In most vibration problems of interest there is a non-zero forcing function  $F(t)$ . This is usually a harmonic force, e.g.:

$$F(t) = F_0 \cos(\omega t) \quad (18)$$

The forced equation of motion becomes:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t) \quad (19)$$

The transient (complementary) solution is still given by equations (13) or (14), but now we need to find a particular solution  $x_p(t)$  satisfying equation (19), which is the response to the forcing function. We can use the method of undetermined coefficients, with

$$x_p(t) = D \cos(\omega t) + E \sin(\omega t) \quad (20)$$

This is also called the **steady-state solution**. Substituting into the O.D.E. we find:

$$(-m\omega^2 D + c\omega E + kD) \cos(\omega t) + (-m\omega^2 E - c\omega D + kE) \sin(\omega t) = F_0 \cos(\omega t) \quad (21)$$

This yields two equations for D and E:

$$\begin{aligned}(k - m\omega^2)D + c\omega E &= F_0 \\ -c\omega D + (k - m\omega^2)E &= 0\end{aligned}\tag{22}$$

from which

$$D = \frac{(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2} F_0 \quad \text{and} \quad E = \frac{c\omega}{(k - m\omega^2)^2 + (c\omega)^2} F_0\tag{23}$$

We can rewrite the particular solution (20) as

$$x_p(t) = X \cos(\omega t - \varphi)$$

where

$$X = \sqrt{D^2 + E^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\tag{24}$$

and

$$\varphi = \text{Tan}^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)\tag{25}$$

Before plotting these results, it is useful to express them non-dimensionally. Recalling that:

$$\begin{aligned}\omega_n &= \sqrt{\frac{k}{m}} = \text{natural frequency of undamped oscillations} \\ c_c &= 2m\omega_n = \text{critical damping} \\ \zeta &= \frac{c}{c_c} = \text{damping factor}\end{aligned}\tag{26}$$

We can write the non-dimensional amplitude and phase as:

$$\frac{Xk}{F_0} = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}; \quad \varphi = \tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right]\tag{27}$$

These equations indicate that the non-dimensional amplitude  $Xk/F_0$  and phase angle are functions only of the frequency ratio  $(\omega/\omega_n)$  and the damping factor  $\zeta$ . The results are plotted in Figure 2.

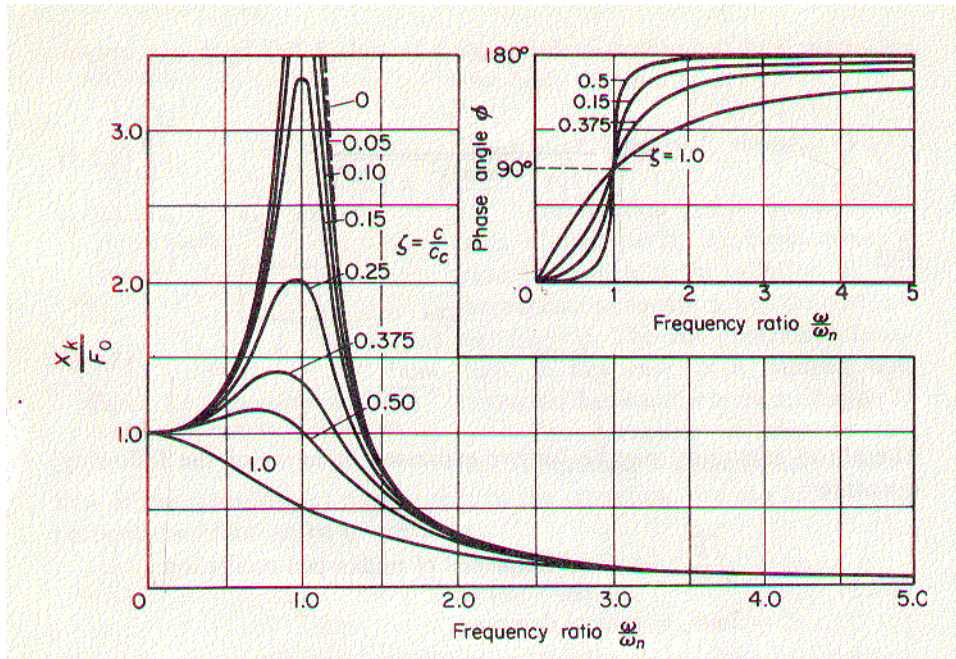


Figure 2. Plot of non-dimensional amplitude and phase angle.

The amplitude of the forced vibration depends on the ratio of the driving frequency  $\omega$  to the natural frequency of vibration  $\omega_n$ . When  $(\omega/\omega_n) \ll 1$ , the amplitude approaches  $X = F_0/k$ , which is equivalent to pulling the spring with a constant force  $F_0$ . When the forcing frequency is very high,  $(\omega/\omega_n) \gg 1$ , the amplitude of response goes to zero, i.e. it is not possible to excite the mass with very high frequency excitation in this case.

The phenomenon of **resonance** occurs when  $\omega \approx \omega_n$ . This leads to high amplitude vibrations, limited only by the damping of the system. To find precisely the frequency of vibration which produces the maximum response, we differentiate equation (27) with respect to  $\omega$ . Setting this derivative to zero, we find

$$\omega_{\max} = \omega_n \sqrt{1 - 2\zeta^2} = \omega_n \sqrt{1 - \frac{c^2}{2km}} \quad (28)$$

Thus for small damping ( $\zeta \rightarrow 0$ ),  $\omega_{\max} \cong \omega_n$ . The phenomenon of resonance is generally something you want to avoid in an engineering structure. If there is a possibility of an exciting force near a natural frequency of vibration, then you should ensure that the damping is high enough to prevent catastrophic amplitudes of vibration.