ME 203: Some Special Exact Differentials **September 17 2002**

When solving first order differential equations which are given in differential form, there are certain combinations of variables which immediately form <u>exact differentials</u>.

Here are some combinations that you may come across:

1. $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

2. $\dfrac{xdy - ydx}{y^2} = -d\left(\dfrac{x}{y}\right)$

3. $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right)$

4. $\dfrac{xdx + ydy}{x^2 + y^2} = d\left(\dfrac{1}{2}\ln\left(x^2 + y^2\right)\right)$

5. $\dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} = d\left(\sqrt{x^2 + y^2}\right)$

6. $\dfrac{xdx - ydy}{\sqrt{x^2 - y^2}} = d\left(\sqrt{x^2 - y^2}\right)$

Illustrative Examples:

1. Solve $xdx + (y - \sqrt{x^2 + y^2})dy = 0.$

<u>Solution</u>: Rewrite as: $\dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} = dy.$

The left hand side is now the same as case 5 above, i.e. $d\left(\sqrt{x^2 + y^2}\right) = dy$.

Integration leads to $\sqrt{x^2 + y^2} = y + c$ or, $y = \dfrac{x^2 - c^2}{2c}$.

(N.B. The original example could also have been solved as a homogeneous equation. Try it!)

2. Solve $\left(x^2 + y^2 + y\right)dx - xdy = 0.$

<u>Solution</u>: It may not be immediately obvious, but dividing through by $\left(x^2 + y^2\right)$ will produce a recognizable form: $dx + \dfrac{ydx - xdy}{x^2 + y^2} = 0$.

This is like case 3 above, so we can write:

$dx - d\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right) = 0$, which can be immediately integrated to yield: $x - \tan^{-1}\left(\dfrac{y}{x}\right) = c.$