ME 203 MIDTERM EXAM SOLUTIONS -

1) CLASSIFY EQUATIONS - 1 point each.

a) $\frac{dy}{dx} - y = 2y^3 + 4$

$\frac{dy}{dx} = y + 2y^3 + 4$

$\frac{dy}{y+2y^3+4} = dx$ $\Rightarrow$ separable 4

$\Rightarrow$ exact 7

$\Rightarrow$ exact with integrating factor 1

b) $x^2 \frac{dy}{dx} - y = -\cos x$

$\frac{dy}{dx} - \frac{y}{x^2} = -\frac{\cos x}{x^2}$ $\Rightarrow$ first order, linear 8

c) $3\frac{dx^*}{dt^*} - 2x^* = 1$ $\Rightarrow$ first order, linear, constant coefficients 2

or separable 4

d) $(2xy^2 + 3x^2)dx + (2x^2y + 4y^3)dy = 0$

$M = 2xy^2 + 3x^2$ $\quad N = 2x^2y + 4y^3$

$\frac{\partial M}{\partial y} = 4xy$ $\quad \frac{\partial N}{\partial x} = 4xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow$ exact 7

e) $(2y^2 + 2y + 4x^2)dx + (2xy + x)dy = 0$

$M = 2y^2 + 2y + 4x^2$ $\quad N = 2xy + x$

$\frac{\partial M}{\partial y} = 4y + 2$ $\quad \frac{\partial N}{\partial x} = 2y + 1$ $\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y+2-(2y+1)}{2xy+x} = \frac{2y+1}{x(2y+1)} = \frac{1}{x}$$

1

$\Rightarrow$ Exact with integrating factor $\mu(x)$

f) $(y^2 - xy)\,dx + x^2\,dy = 0$

$$x^2\,dy = (xy - y^2)\,dx$$

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{xy}{x^2} - \frac{y^2}{x^2} = \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2$$ $\Rightarrow$ first order homogeneous 6

$\rightarrow$ or Bernoulli 3

g) $\frac{dy}{dx} + \frac{y}{x-2} = 5(x-2)\,y^{1/2}$ $\Rightarrow$ Bernoulli equation 3

h) $\frac{d^2y}{dx^2} - 2xy = x$ $\Rightarrow$ second order, non-homogeneous 9

i) $3\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 7y = 0$ $\Rightarrow$ second order linear, with constant coefficients 5

j) $x\frac{dy}{dx} = y\sin x$

$\frac{dy}{dx} = y\frac{\sin x}{x}$ $\quad$ $\frac{dy}{y} = \frac{\sin x}{x}\,dx$ $\Rightarrow$ separable 4

$\Rightarrow$ first order linear 8

2) a) $\frac{dy}{dx} + y\tan x + \sin x = 0$ $\quad$ $y(0) = \pi$

linear, first order <u>not separable</u>

put in standard form $\frac{dy}{dx} + P(x)\,y = Q(x)$

$P(x) = \tan x$ $\quad$ $Q(x) = -\sin x$

integrating factor $\mu = \exp\left(\int P(x)\,dx\right)$

$$\int P(x)\,dx = \int \tan x\,dx$$

$$= \ln \sec x \quad \text{(Schaum's, pg 65)}$$

$$\mu = \exp(\ln|\sec x|)$$

$$= \sec x$$

apply integrating factor

$$\frac{d}{dx}\left[\sec x \cdot y\right] = -\sec x \cdot \sin x$$

$$= \tan x$$

$$\int \frac{d}{dx}\left[\sec x \cdot y\right] dx = -\int \tan x \, dx$$

$$\sec x \cdot y = -\ln(\sec x) + C$$

$$\frac{y}{\cos x} = \ln(\cos x) + C$$

$$y = \cos x \cdot \ln(\cos x) + C\cos x$$

apply initial condition $y(0) = \pi$

$$y(0) = 0 + C = \pi \quad \therefore \quad C = \pi$$

$$\boxed{y = \cos x \cdot \ln(\cos x) + \pi \cos x}$$

2) b) $(y^3 + 4e^x y)\,dx + (2e^x + 3y^2)\,dy = 0$

$M = y^3 + 4e^x y$ $\qquad N = 2e^x + 3y^2$

$\frac{\partial M}{\partial y} = 3y^2 + 4e^x$ $\qquad \frac{\partial N}{\partial x} = 2e^x$ $\qquad \therefore$ not exact

$$\left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right) = \frac{3y^2 + 4e^x - 2e^x}{2e^x + 3y^2} = \frac{3y^2 + 2e^x}{3y^2 + 2e^x} = 1$$

$\therefore$ integrating factor function of x only.

$$\mu = \exp\left(\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right) dx\right) = \exp \int dx$$

$$\therefore \mu = e^x$$

$$F = \int \mu \cdot M \, dx$$

$$= \int e^x (y^3 + 4e^x y)\, dx = \int e^x y^3 + 4e^{2x} y \, dx$$

$$= y^3 e^x + 2ye^{2x} + g(y)$$

determine integration coefficient $g(y)$

$$\frac{\partial F}{\partial y} = 3y^2 e^x + 2e^{2x} + g' = \mu \cdot N$$

$$= e^x(2e^x + 3y^2)$$

$$= 2e^{2x} + 3y^2 e^x$$

$$\therefore g' = 0$$

$$g = \text{constant} = C$$

$$\boxed{F = y^3 e^x + 2ye^{2x} + C}$$

2) c) $\dfrac{dy}{dx} = \dfrac{10-2x}{y^3 - y - 6}$ $\quad y(2) = -1$

separable $\quad y^3 - y - 6 \, dy = 10 - 2x \, dx$

integrate both sides $\quad \dfrac{y^4}{4} - \dfrac{y^2}{2} - 6y = 10x - x^2 + C$

substitute initial condition $\quad y(2) = -1$

$$\frac{(-1)^4}{4} - \frac{(-1)^2}{2} - 6(-1) = 10(2) - (2)^2 + C$$

$$\frac{1}{4} - \frac{1}{2} + 6 = 20 - 4 + C$$

$$C = 5\tfrac{3}{4} + 4 - 20$$

$$= -10\tfrac{1}{4} = -\frac{41}{4}$$

$$\boxed{\frac{y^4}{4} - \frac{y^2}{2} - 6y = 10x - x^2 - \frac{41}{4}}$$

2) d) $2xy\,\frac{dy}{dx} = 4x^2 + 3y^2$

$$\frac{dy}{dx} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x}$$

- first order homogeneous - let $v = \frac{y}{x}$

$y = v \cdot x$

$\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v$

$$\frac{dv}{dx} \cdot x + v = \frac{2}{v} + \frac{3}{2} v$$

$$\frac{dv}{dx} = \frac{\left[\frac{2}{v} + \frac{1}{2} v\right]}{x} \Leftarrow \text{separable}$$

$$\frac{2v}{2v} \cdot \frac{dv}{\frac{2}{v} + \frac{v}{2}} = \frac{dx}{x} \Rightarrow \frac{2v\,dv}{v^2+4} = \frac{dx}{x}$$

$$= 2\int \frac{v\,dv}{v^2+4} = \int \frac{dx}{x}$$

(from Schaum's pg. 72)

$$= 2 \cdot \frac{1}{2} \ln(v^2+4) = \ln x + C_1$$

$$e^{\ln|v^2+4|} = e^{\ln x + C_1}$$

$$v^2+4 = C_2 \cdot x$$

$$v^2 = C_2 \cdot x - 4$$

$$\boxed{v = \pm\sqrt{C_2 x - 4}}$$

2) e) $\frac{dy}{dx} = (2x+y-1)^2$ use transformation

$v = 2x + y - 1$

$y = v - 2x + 1$

$\frac{dy}{dx} = \frac{dv}{dx} - 2$

$$\frac{dv}{dx} - 2 = v^2$$

$$\frac{dv}{dx} = v^2 + 2 \Leftarrow \text{separable}$$

$$\int \frac{dv}{v^2+2} = \int dx$$

$$\frac{1}{\sqrt{2}} \arctan \frac{v}{\sqrt{2}} = x + C_1 \quad \text{(Schaum's pg. 72)}$$

take tan of both sides

$$\frac{v}{\sqrt{2}} = \tan\left(\sqrt{2}x + C_2\right)$$

$$v = \sqrt{2}\tan\left(\sqrt{2}x + C_2\right) = 2x + y - 1$$

$$\boxed{y = \sqrt{2}\tan\left(\sqrt{2}x + C_2\right) - 2x + 1}$$

2) f) $2\dfrac{d^2x}{dt^2} - 8\dfrac{dx}{dt} + 8x = 0 \qquad x(0) = 1, \quad x'(0) = -2$

- linear, 2nd order constant coefficients

substitute $y = e^{rt} \quad y' = re^{rt} \quad y'' = r^2e^{rt}$

$$2r^2e^{rt} - 8re^{rt} + 8e^{rt} = 0$$

$$e^{rt}(2r^2 - 8r + 8) = 0$$

check discriminant $b^2 - 4ac = 64 - 4(2)(8) = 0$

$\therefore$ real, repeated roots

$$r = \frac{-b}{2a} = \frac{8}{2(2)} = 2$$

standard form of solution

$$x = C_1e^{rt} + C_2te^{rt}$$

$$x = C_1e^{2t} + C_2te^{2t}$$

substitute boundary conditions

$$x(0) = C_1 = 1$$

$$x' = \frac{d}{dt}\left[e^{2t} + C_2 t e^{2t}\right] = C_2\left[e^{2t} + 2te^{2t}\right] + 2e^{2t}$$

e^{2t}

$$x'(0) = C_2 + 2 = -2 \qquad C_2 = -4$$

?

$$\boxed{x = e^{2t} - 4te^{2t}}$$

3) - hailstone falling from rest $\quad \rho = 1\ kg/m^3$

$r(t=0) = 0$

- radius increase $\quad r(t) = k \cdot t \quad v(t=0) = 0$

- neglect drag, buoyant forces.

① IDENTIFY VARIABLES $\quad$ dependent ⇒ momentum $= m \cdot v$ $[kg \cdot \frac{m}{s}]$

1 pt. $\quad$ independent ⇒ time $[s]$

② FUNDAMENTAL LAWS.

Conservation of Momentum $\quad \frac{dp}{dt} = \sum \text{Forces}$

1 pt

Newton's Second Law $\quad F = m \cdot g$ ⇐ gravity is the only force acting on the body

1 pt $\quad$ ∴ ODE $\quad \frac{d}{dt}(m \cdot v) = m \cdot g$

mass of hailstone $\quad m = \rho \cdot V \qquad V$ ⇒ volume sphere $= \frac{4}{3}\pi r^3$

$$m = \rho \cdot \frac{4}{3}\pi r^3$$

1 pt $\quad$ radius $= f(t) = k \cdot t \quad ∴ \ m(t) = \rho \cdot \frac{4}{3}\pi k^3 t^3$

substitute into ODE $\quad \frac{d}{dt}\left(\frac{4}{3}\rho \pi k^3 t^3 \cdot v\right) = \rho \cdot \frac{4}{3}\pi k^3 t^3 \cdot g$

1pt simplify $\frac{d}{dt}\left(t^3 \cdot v\right) = t^3 \cdot g$

③ SOLVE ODE $\Rightarrow$ solve for v

chain rule $3t^2 \cdot v + t^3 \cdot \frac{dv}{dt} = t^3 \cdot g$ ✓

1pt re-arrange to standard form $\frac{dv}{dt} + \frac{3}{t} v = g$

first order, linear equation $\Rightarrow$ NOT separable.

solve using integrating factor

$$\mu = \exp\left(\int P(t) \cdot dt\right) \qquad P(t) = \frac{3}{t}$$

$$\int P(t)\,dt = 3 \ln t$$

3pt $$\exp(3 \ln t) = \exp(\ln t^3) = t^3$$

apply integrating factor and solve

$$\frac{d}{dt}\left[t^3 \cdot v\right] = t^3 \cdot g$$

integrate both sides wrt t

$$t^3 \cdot v = \frac{t^4}{4} \cdot g + C \text{ ✓}$$

$$v = \frac{g \cdot t}{4} + C$$

Substitute initial condition $v(0) = 0$

1pt $$v(0) = 0 + C \quad \therefore \quad C = 0$$

④ interpret results

$$\boxed{v(t) = \frac{g \cdot t}{4} \quad \left[\frac{m}{s}\right]}$$

4) Cake removed from oven $T(0) = 100\,°C$

Cooled at room temperature $T_{room} = 20\,°C$

① IDENTIFY VARIABLES dependent ⇒ temperature [°C]

1pt. independent ⇒ time [min]

② FUNDAMENTAL LAWS

Conservation of Energy

no energy input into system

1pt $$\frac{dT}{dt} = \frac{1}{m \cdot c_p}(Q_{in} - Q_{out}) \quad \text{where } Q_{in} = 0$$

Newtons Law of Cooling (given in problem)

$$Q_{out} = k \cdot (T - T_{room}) \quad \text{where } k = \text{constant}$$

1pt $$\frac{dT}{dt} = \frac{-k}{m \cdot c_p}(T - T_{room})$$

③ SOLVE ODE

$$\frac{dT}{dt} = \frac{-k}{m \cdot c_p}(T - T_{room}) \Leftarrow \text{separable.}$$

$$\int \frac{dT}{T - T_{room}} = \int \frac{-k}{m \cdot c_p}\, dt$$

$$\ln(T - T_{room}) = \frac{-k}{m \cdot c_p} t + C_1$$

$$\exp(\ln(T - T_{room})) = e^{\frac{-k}{m \cdot c_p} t + C_1}$$

$$T - T_{room} = C_2 e^{\frac{-k}{m \cdot c_p} t}$$

2pt. $$T = C_2 e^{\frac{-k}{m \cdot c_p} t} + T_{room}.$$

substitute initial condition $T(0) = 100°C$

$T_{room} = 20°C$

$$T(0) = C_2 + 20°C = 100°C$$

1pt.

$$\therefore C_2 = 80°C$$

$$T(t) = 80°C \cdot e^{\frac{-k}{mc_p}t} + 20°C$$

solve for constant $k/(m \cdot c_p)$ using intermediate condition

$$T(30) = 60°C$$

$$T(30) = 80 \cdot e^{\frac{-k}{mc_p} \cdot 30} + 20 = 60$$

$$e^{\frac{-k}{mc_p} \cdot 30} = \frac{1}{2}$$

2pt.

$$-\frac{k}{m \cdot c_p} \cdot 30 = \ln\left(\frac{1}{2}\right)$$

$$\frac{k}{m \cdot c_p} = -\frac{1}{30} \cdot \ln\left(\frac{1}{2}\right) \cong 0.0231$$

1pt

$$\boxed{\therefore T(t) = 80 \cdot e^{-0.0231t} + 20}$$ ⇐ solution part a)

(4) INTERPRET RESULTS.

Find t when $T = 38°C$

$$80 \cdot e^{-0.0231t} + 20 = 38$$

$$e^{-0.0231t} = \frac{18}{80}$$

$$0.0231t = -\ln\left(\frac{18}{80}\right)$$

$$t = \frac{-1}{0.0231} \cdot \ln\left(\frac{18}{80}\right)$$

1pt

$$\boxed{t = 64.6 \text{ minutes.}}$$ ⇐ solution part b)

5) spring - mass - dashpot system

$m = 10$ kg
$k = 250$ N/m
$c = 60$ N·s/m

INITIAL CONDITIONS
$x(0) = 0.3$ m
$x'(0) = -0.1$ m/s

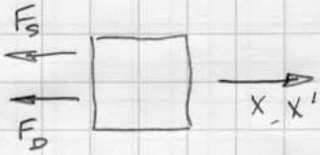
① IDENTIFY VARIABLES

dependent x [m]

independent t [s]

② FUNDAMENTAL LAWS.

Force Balance



$F_s = k \cdot x$

$F_D = c \cdot \frac{dx}{dt}$

Newtons 2nd Law

$F = ma$ where $a = \frac{d^2x}{dt^2}$

$$m \cdot \frac{d^2x}{dt^2} = -F_s - F_D$$

$$m \cdot \frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

③ SOLVE ODE

2nd order, linear, constant coefficient

check discriminant $b^2 - 4ac = c^2 - 4(m)(k)$

$$= (60)^2 - 4(10)(250)$$

$$= -6400 \quad \therefore \text{ complex roots}$$

solve for roots

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-60 \pm \sqrt{-6400}}{20}$$

$$= -\frac{60}{20} \pm i\frac{80}{20} = -3 \pm i4$$

standard form of solution for complex roots

$$x = e^{\alpha t}\left(C_1 \cos\beta t + C_2 \sin\beta t\right)$$

where $r = \alpha \pm i\beta$

$$\therefore \quad \alpha = -3 \qquad \beta = 4$$

$$x = e^{-3t}\left(C_1 \cos 4t + C_2 \sin 4t\right)$$

substitute boundary conditions

$$x(0) = C_1 \cos(0) = 0.3 \quad \therefore \quad C_1 = 0.3$$

$$x' = 0.3 \frac{d}{dt}\left(e^{-3t} \cdot \cos 4t\right) + C_2 \frac{d}{dt}\left(e^{-3t} \sin 4t\right)$$

$$= 0.3\left[-3e^{-3t}\cos 4t - 4e^{-3t}\sin 4t\right]$$

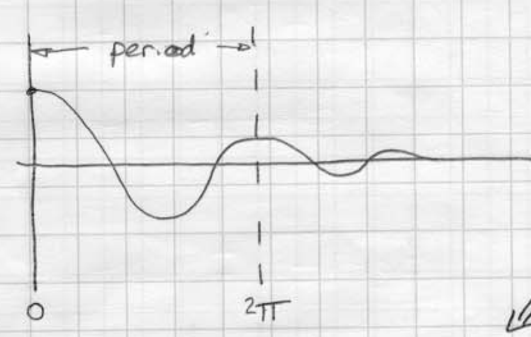
$$+ C_2\left[-3e^{-3t}\sin 4t + 4e^{-3t}\cos 4t\right]$$

$$x'(0) = (0.3)(-3) + C_2(4) = -0.1$$

$$C_2 = \frac{-0.1 + 0.9}{4} = \frac{0.8}{4} = 0.2$$

$$\boxed{x = e^{-3t}\left(0.3 \cos 4t + 0.2 \sin 4t\right)}$$ ⇐ solution part a)

4 INTERPRET RESULTS.



$$\text{Frequency} = \frac{1}{\text{period}}$$

period $4t = 2\pi$

$$t = \frac{\pi}{2} \text{ s}$$

solution part b)

$$\boxed{\therefore \text{ Frequency} = \frac{2}{\pi} \cong 0.637 \text{ Hz}}$$