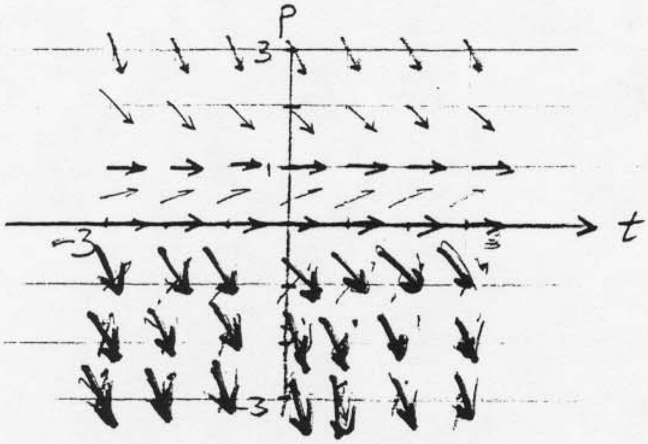
ME203 – Midterm Exam Solutions (10 June 2002)

#1 $\frac{dp}{dt} = p(1-p)$

(a) For equilibrium solution $\frac{dp}{dt} = 0 \Rightarrow p = 0$ or $p = 1$ ← 3 marks

(b)



Direction Field
$|p| \leq 3$
$|t| \leq 3$

← 4 marks

(c)

$$\frac{dp}{p(1-p)} = dt \qquad \left[\frac{1}{p(1-p)} = \frac{1}{p} + \frac{1}{1-p}\right]$$

$$\frac{dp}{p} + \frac{dp}{1-p} = dt$$

$$\ln|p| - \ln|(1-p)| = t + c_1$$

$$\ln\left|\left(\frac{p}{1-p}\right)\right| = t + c_1$$

$$\left|\left(\frac{p}{1-p}\right)\right| = e^{c_1} e^{t}$$

$$\frac{p}{1-p} = c_2 e^{t} \qquad (c_2 = \pm e^{c_1})$$

$$p(1 + c_2 e^{t}) = c_2 e^{t}$$

$$p = \frac{c_2 e^{t}}{c_2 e^{t} + 1} = \frac{1}{1 + c_3 e^{-t}} \qquad (c_3 = 1/c_2)$$

Since $p(0) = 3 \implies 3 = \dfrac{1}{1 + c_3}$

$$c_3 = -\frac{2}{3}$$

$$\boxed{p(t) = \frac{1}{1 - \frac{2}{3} e^{-t}}}$$ ← ~~7~~ marks

$\lim_{t \to \infty} p(t) = \dfrac{1}{1-0} = 1$, in this case ← 1 mark

#2 $$(x^2+y^2)\frac{dy}{dx} = xy$$

$$\frac{dy}{dx} = \frac{y/x}{1+(y/x)^2} \implies \text{homogeneous}$$

let $v = y/x \Rightarrow \frac{dy}{dx} = v + xv'$

$$v + xv' = \frac{v}{1+v^2}$$

$$xv' = \frac{v}{1+v^2} - v = \frac{-v^3}{1+v^2}$$

Separable
$$\frac{1+v^2}{v^3}\,dv = -\frac{dx}{x}$$

$$\left(\frac{1}{v^3} + \frac{1}{v}\right)dv = -\frac{dx}{x}$$

$$\frac{v^{-2}}{-2} + \ln|v| = -\ln|x| + c_1$$

$$\ln\left|\frac{y}{x}\right| - \frac{1}{2}\left(\frac{x}{y}\right)^2 + \ln|x| = c_1$$

$$\ln|y| - \frac{1}{2}\left(\frac{x}{y}\right)^2 = c_1 \leftarrow \text{8 marks}$$

For $y(1) = 1 \implies$ $$0 - \frac{1}{2} = c_1$$

$$\therefore \quad \ln|y| - \frac{1}{2}\left(\frac{x}{y}\right)^2 = -\frac{1}{2}$$ $\leftarrow$ 3 marks

#3 Write as:

$$\frac{dy}{dt} - \left(\frac{t}{1-t^2}\right) y = t$$

(a) Linear equation — of form $\frac{dy}{dt} + p(t)\, y = q(t)$] 2 marks

(b) Integrating factor $\mu(t) = \exp \int \frac{-t}{(1-t^2)} dt$

$$= \exp\left(\tfrac{1}{2}\ln(1-t^2)\right)$$
$$= \sqrt{1-t^2}$$

$$\therefore \quad \frac{d}{dt}\left(\sqrt{1-t^2}\, y\right) = t\sqrt{1-t^2}$$

$$\sqrt{1-t^2}\, y = \int t\sqrt{1-t^2}\, dt + c_1$$

$$= -\frac{1}{3}(1-t^2)^{3/2} + C_1$$

$$\boxed{y = -\frac{1}{3}(1-t^2) + \frac{C_1}{\sqrt{1-t^2}}}$$ ← 8 marks

For initial condition $y(0) = 2 \Rightarrow \quad 2 = -\frac{1}{3} + C_1$

$$C_1 = \frac{7}{3}$$] 3 marks

(c) Write as:

$$\frac{dy}{dt} = f(t,y) = t + \left(\frac{t}{1-t^2}\right) y$$

Since $f(t,y)$ is not defined at $t=1$ (not continuous)
$\therefore$ we are not guaranteed solutions at $t=1$ exist] 2 marks

#4 $\quad (4x^2-2y)\,dx - x\,dy = 0$

(a) $\dfrac{\partial M}{\partial y} = -2 \quad , \quad \dfrac{\partial N}{\partial x} = -1 \Rightarrow$ Not exact (2 marks)

However $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{-1}{-x} = \dfrac{1}{x} \Rightarrow$ fn (x) only

We let $\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$ (3 marks)

Then $\quad (4x^3 - 2xy)\,dx - x^2\,dy = 0$

Now Exact since: $\dfrac{\partial M}{\partial y} = -2x \quad \dfrac{\partial N}{\partial x} = -2x$

(b) $\quad F(x,y) = \displaystyle\int^x (4x^3 - 2xy)\,dx + g(y)$

$$= x^4 - x^2 y + g(y)$$

$$\frac{\partial F}{\partial y} = -x^2 + g'(y) = -x^2$$

$$\therefore g'(y) = 0$$

General Solution: $F(x,y) = x^4 - x^2 y = c_1$

$$\boxed{y = x^2 - \frac{c_1}{x^2}}$$ (5 marks)

(c) Write as: $\dfrac{dy}{dx} = \dfrac{4x^2-2y}{x} = -\dfrac{2}{x}y + 4x$

$\therefore \quad \dfrac{dy}{dx} + \dfrac{2}{x}y = 4x$ (2 marks) $\quad p(x) = \frac{2}{x}$; $q(x) = 4x$

Integrating factor: $\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$ (3 marks)

$\therefore \quad \dfrac{d}{dx}\left(x^2 y\right) = 4x^3$

$x^2 y = x^4 + c_2$

$$\boxed{y = x^2 + \frac{c_2}{x^2}}$$ (5 marks)

Same result with $c_2 = -c_1$

#5 $y'' + y' - 3y = 0$ let $y = e^{rt}$

Characteristic Equation $r^2 + r - 3 = 0$

$r = \dfrac{-1 \pm \sqrt{13}}{2}$

$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{13}$

5 marks

$y(t) = c_1 e^{-\frac{1+\sqrt{13}}{2}t} + c_2 e^{-\frac{1-\sqrt{13}}{2}t}$

$y(0) = 0 \Rightarrow \quad c_1 + c_2 = 0 \qquad c_1 = -c_2$

$y'(0) = 1 \Rightarrow \quad c_1\left(\dfrac{-1+\sqrt{13}}{2}\right) + c_2\left(\dfrac{-1-\sqrt{13}}{2}\right) = 1$

$c_1\left(-\dfrac{1+\sqrt{13}}{2}\right) + c_1\left(\dfrac{1+\sqrt{13}}{2}\right) = 1$

$c_1\left(\sqrt{13}\right) = 1$

5 marks

$$\therefore \quad y(t) = \frac{\sqrt{13}}{13} \exp\left(\frac{-1+\sqrt{13}}{2} t \right) - \frac{\sqrt{13}}{13} \exp\left(\frac{-1-\sqrt{13}}{2} t \right)$$

#6 $\frac{dM}{dt} = \dot{M}_{in} - \dot{M}_{out}$ [Mass Conservation Law]

Mass of Methane in room: $M = 200\, C(t)$ [g] (2)

Rate of addition $\dot{m}_{in} = 50$ g/min (2)

Rate of removal: $\dot{m}_{out} = \frac{50\, C(t)}{60}$ g/min (2)

$$\therefore \quad 200 \frac{dc}{dt} = 50 - \left(\frac{50}{60}\right) C$$

$$\frac{dc}{dt} = \tfrac{1}{4} - \tfrac{1}{240} c$$

$$\frac{dc}{dt} = \frac{60-c}{240}$$

$$\frac{dc}{60-c} = \frac{t}{240}$$

$$\frac{dc}{dt} = \frac{1}{4} - \frac{1}{240} C$$

$$\frac{dc}{dt} + \frac{1}{240} C = \frac{1}{4}$$

(4)

Integrating factor $\mu(t) = e^{\int \frac{1}{240} dt} = e^{\frac{t}{240}}$

$$\therefore \quad \frac{d}{dt}\left(e^{\frac{t}{240}} C\right) = \frac{1}{4} e^{\frac{t}{240}}$$

$$e^{\frac{t}{240}} C = 60\, e^{\frac{t}{240}} + C_1$$

$$C(t) = 60 + C_1 e^{-t/240}$$ (4)

Initial condition $C(0) = 0 \Rightarrow 60 + C_1 = 0$

$C_1 = -60$ (2)

$$\therefore \quad \boxed{C(t) = 60\left(1 - e^{-t/240}\right)}$$

(b) $\lim_{t \to \infty} C(t) = 60\,(1 - 0) = 60 \text{ g/m}^3$ (4)

(c) let $C(\tau) = 60\left(1 - e^{-\tau/240}\right) = 35$

$$1 - e^{-\tau/240} = \frac{7}{12}$$

$$e^{-\tau/240} = \frac{5}{12}$$

(6)

$$\tau = 240 \ln\left(\frac{12}{5}\right) = 210 \text{ mins.}$$

$\therefore$ LEL is reach after 210 mins (3.5 hrs)

Don't light a cigarette!

#7

(a) $$M\frac{dV}{dt} = -k_1V - k_2V^2$$ (3) Bernoulli-type equation!

(b) let $y = \frac{1}{V}$ $\quad \frac{dV}{dt} = -\frac{1}{y^2}\frac{dy}{dt}$ (2)

$$\therefore \quad -\frac{M}{y^2}\frac{dy}{dt} = -\frac{k_1}{y} - \frac{k_2}{y^2}$$

$$\frac{dy}{dt} = \frac{k_1}{M}y + \frac{k_2}{M} \quad (4)$$ Integrating factor $\mu(t) = e^{-\frac{k_1}{M}t}$

$$\frac{d}{dt}\left(e^{-\frac{k_1}{M}t}y\right) = \frac{k_2}{M}e^{-\frac{k_1}{M}t}$$

$$e^{-\frac{k_1}{M}t}y = -\frac{k_2}{k_1}e^{-\frac{k_1}{M}t} + C_1$$

$$y = C_1 e^{+\frac{k_1}{M}t} - \frac{k_2}{k_1}$$

$$V(t) = \frac{1}{y} = \frac{1}{C_1 e^{+\frac{k_1}{M}t} - \frac{k_2}{k_1}} \quad (4)$$

$$V(0) = V_0 = \frac{1}{C_1 - \frac{k_2}{k_1}} \quad \therefore \quad C_1 = \frac{1}{V_0} + \frac{k_2}{k_1}$$

$$\therefore \quad V(t) = \frac{1}{\frac{1}{V_0}\left(e^{k_1/m\, t}\right) + \frac{k_2}{k_1}\left(e^{\frac{k_1}{m}t} - 1\right)} = \frac{1}{e^{\frac{k_1}{m}t}\left(\frac{1}{V_0} + \frac{k_2}{k_1}\right) - \frac{k_2}{k_1}}$$

$$\boxed{V(t) = \frac{V_0\, e^{-\frac{k_1}{m}t}}{1 + \frac{k_2}{k_1} V_0 \left(1 - e^{-\frac{k_1}{m}t}\right)}} \qquad (4)$$

(c) When $k_2 = 0$

$$V(t) = V_0\, e^{-\frac{k_1}{m}t} \qquad \text{(exponential decay)} \qquad (5)$$

When $k_1 = 0 \Rightarrow$ need to re-evaluate the original solution, because above expression not defined