ME203 – Ordinary Differential Equations
Spring 2002 Midterm Examination

Professor Rob Macdonald **10 June, 2002**

Instructions:

1. Permitted aids: non-programmable scientific calculator and a hand-written equation list consisting of one side of 8½”x11” paper. Integral tables are appended to this exam.
2. Answer the first 5 questions plus one of either question 6 or 7. If you solve all seven, the best six count.
3. Do not spend more than 15-20 minutes on any one question!
4. Clear, systematic solutions are required. Part marks will be rewarded for part answers, provided that I can follow your methodology.
5. The time limit is 2 hours.
6. Each question is worth marks indicated.

---

1. The logistic equation describing the population p(t) of a certain species is given by:

$$\frac{dp}{dt} = p(1-p)$$

(a) By inspecting the differential equation, what are the equilibrium solutions to this problem?
(b) Sketch the direction field for the solution for $-3 \le t \le 3$ and $-3 \le p \le 3$.
(c) Solve the equation for the initial condition $p(0) = 3$. What is $\lim_{t\to\infty} p(t)$ in this case?

2. Find the general solution to the equation:

$$(x^2 + y^2)\frac{dy}{dx} = xy$$

Evaluate the arbitrary constant in your solution for initial condition y(1) = 1.

3. Consider the following first order differential equation:

$$(1-t^2)\frac{dy}{dt} - ty = t(1-t^2).$$

(a) What type of equation is this? Find the general solution.
(b) Find the unique solution for the initial condition y(0) = 2.
(c) Show that there is no solution for the initial condition y(1) = 2. Why is this not a violation of the existence theorem?

4. Consider the following differential equation:

$$(4x^2 - 2y)dx - xdy = 0.$$

   (a) Show that it is not exact, but that it is possible to find an integrating factor μ(x) which makes it exact.
   (b) Solve the resulting equation.
   (c) Show that the original equation can also be written as a linear ODE of the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

   Find the integrating factor in this case and show that the same solution as part (b) is found.

5. Find the solution to the second order equation:

$$y'' + y' - 3y = 0,$$

which satisfies the initial conditions y(0) = 0, y′(0) = 1.

Choose Only One of the Following Two Questions:

6. A stove top burner leaks methane gas at a rate of 50 g/min into a room of volume 200 m^3. Fresh air enters the room at a rate of 50 m^3/hour and the methane-polluted air leaves at the same rate. In your analysis, assume that the methane gas is uniformly mixed throughout the room at all times.

   (a) Determine the concentration of methane gas as a function of time (in minutes) if the gas leak begins at time t = 0, and the room air is initially free of methane.
   (b) What is the concentration in the room as $t \to \infty$?
   (c) How long will it take before the concentration of methane reaches the lower explosion limit (LEL) of 35 g/m^3?

7. If the fuel to a car engine is shut off, the car will eventually come to a rolling stop, due to wind resistance and rolling friction. Assume that the rolling friction is given by $F_f = k_1V$ and the wind resistance is given by $F_w = k_2V^2$ where V is the speed of the car.

   (a) Write down the equation for the velocity of the car. Assume that car has mass M.
   (b) Solve the resulting equation if $V(0) = V_0$ [Hint: Define a new variable $y = V^{-1}$].
   (c) What is the solution if $k_2 = 0$?