ME 203 — Midterm

(15 June, 2000)

ME 203 - Midterm Solutions

#1 $$m\ddot{x} + c\dot{x} + kx = 0$$

Characteristic equation: $mr^2 + cr + k = 0$

(a) $\left.\begin{array}{l} m=1 \\ c=3 \\ k=2 \end{array}\right\} \Rightarrow$ $r^2 + 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r_1 = 2, \ r_2 = 1$

$$x(t) = c_1 e^{2t} + c_2 e^{t}$$

Initial Conditions $x(0) = 1 \rightarrow c_1 + c_2 = 1$

$\dot{x}(0) = 0 \Rightarrow 2c_1 + c_2 = 0$

$\therefore \ c_1 = -1 \quad c_2 = 2$

$$\boxed{x(t) = -e^{2t} + 2e^{t}}$$

(b) $\left.\begin{array}{l} m=1 \\ c=2 \\ k=2 \end{array}\right\} \Rightarrow$ $r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{4-8}}{2}$$

$r_1 = -1 + i \quad r_2 = -1 - i$

$$x(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

$$\dot{x}(t) = -c_1 e^{-t}\cos t - c_1 e^{-t}\sin t - c_2 e^{-t}\sin t + c_2 e^{-t}\cos$$

$$x(0) = 1 \implies c_1 = 1$$
$$\dot{x}(0) = 0 \implies -c_1 + c_2 = 0 \implies c_2 = 1$$

$$\boxed{x(t) = e^{-t}(\cos t + \sin t)}$$

#2 $$y' + y\left(\frac{\sin x}{\cos x}\right) = 1$$

Let $$\mu(x) = \exp\left(\int \frac{\sin x}{\cos x}\,dx\right) = \exp(-\ln \cos x)$$
$$= \frac{1}{\cos x}$$

Rewrite as $$\frac{d}{dx}\left(\frac{y}{\cos x}\right) = \frac{1}{\cos x}$$

$$\frac{y}{\cos x} = \int^x \sec x\,dx + c_1$$

$$y = \cos x \ln|\sec x + \tan x| + c_1 \cos x$$

$$y(0) = 1 \ln|1 + 0| + c_1 = c_1$$

$$\therefore c_1 = 1$$

$$\boxed{y(x) = \cos x\,(1 + \ln|\sec x + \tan x|)}$$

#3 $y' - 2xy = 1$

Let $\mu(x) = \exp\left(-\int 2x\,dx\right) = e^{-x^2}$

Rewrite as: $\frac{d}{dx}\left(ye^{-x^2}\right) = e^{-x^2}$

$$ye^{-x^2} = \int_{c_1}^{x} e^{-u^2}\,du$$

$u \equiv$ "dummy variable"

$$\boxed{y = e^{x^2}\int_{c_1}^{x} e^{-u^2}\,du}$$

$$y(x_0) = e^{x_0^2}\int_{c_1}^{x_0} e^{-u^2}\,du = 0$$

$$\therefore \quad c_1 = x_0$$

$$\boxed{y(x) = e^{x^2}\int_{x_0}^{x} e^{-u^2}\,du}$$

\#4 $$\frac{dy}{dx} = \frac{x}{x^2y+y^3}$$

$$\underset{M}{\underset{\downarrow}{x}}\,dx - \underset{N}{\underset{\downarrow}{(x^2y+y^3)}}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -2xy$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ not exact

Multiply by $\mu(y)$ then

$$\underset{M}{\underset{\downarrow}{x\mu(y)}}\,dx - \underset{N}{\underset{\downarrow}{(x^2y+y^3)\mu(y)}}\,dy = 0$$

$$\frac{\partial M}{\partial y} = x\mu'(y) \qquad \frac{\partial N}{\partial x} = -2xy\mu(y)$$

For exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \frac{\mu'(y)}{\mu(y)} = -2y$

$$\ln|\mu(y)| = -y^2$$

$$\boxed{\mu(y) = e^{-y^2}}$$

Find solution $U(x,y) =$ constant, such that

$$\frac{\partial U}{\partial x} = M = x e^{-y^2} \Rightarrow U(x,y) = \frac{x^2}{2} e^{-y^2} + g(y)$$

$$\frac{\partial U}{\partial y} = -x^2 y e^{-y^2} + g'(y) = -x^2 y e^{-y^2} + y^3 e^{-y^2}$$

$$\therefore g'(y) = y^3 e^{-y^2}$$

$$g(y) = \int y^3 e^{-y^2} dy$$

$$= y^2 \frac{e^{-y^2}}{-2} + \int y e^{-y^2} dy$$

$$= -\frac{1}{2} y^2 e^{-y^2} - \frac{1}{2} e^{-y^2}$$

$$\therefore U(x,y) = \frac{e^{-y^2}}{2}\left[x^2 - y^2 - 1 \right] = c_1$$

When $x=1, y=1$

$$\Rightarrow \frac{e^{-1}}{2}(-1) = c_1$$

$$\therefore \boxed{\frac{e^{-y^2}}{2}\left(x^2 - y^2 - 1 \right) = \frac{-1}{2e}}$$

#5 $$\frac{dy}{dx} = \frac{4y^2 - x^4}{4xy}$$

Let $y(x) = x v(x) \Rightarrow \frac{dy}{dx} = v + x v'$

$$v + x v' = \frac{4x^2 v^2 - x^4}{4x^2 v} = \frac{4v^2 - x^2}{4v}$$

$$x v' = \frac{4v^2 - x^2}{4v} - \frac{4v^2}{4v} = \frac{-x^2}{4v}$$

$$4v\, dv = -x\, dx$$

$$2v^2 = -\frac{x^2}{2} + C_1$$

$$2\left(\frac{y^2}{x^2}\right) = -\frac{x^2}{2} + C_1$$

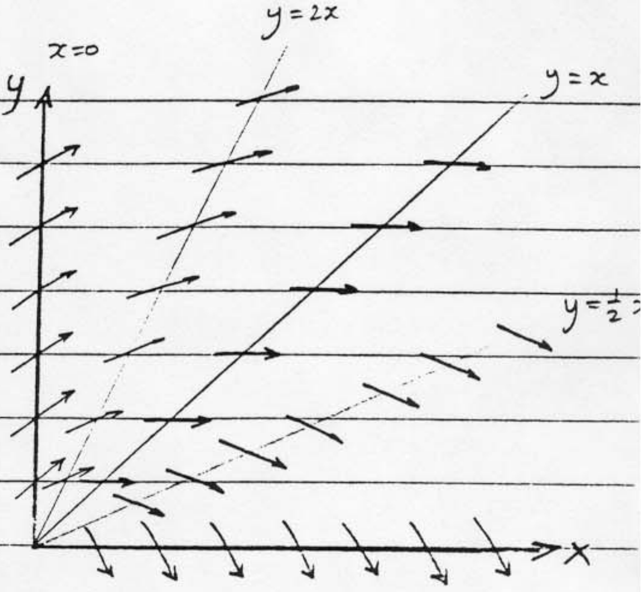
$$\boxed{y^2 = -\frac{1}{4} x^4 + \frac{1}{2} C_1 x^2}$$

$y(0) = 0 \Rightarrow 0 = 0$

$\therefore$ All solutions of form $y^2 = -\frac{1}{4}x^4 + \frac{1}{2}C_1 x^2$ pass through origin.

This does not violate uniqueness theorem since $\frac{dy}{dx} = \frac{0}{0}$ is undefined at origin.

#6 $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$



(a) When $x=0$ $\dfrac{dy}{dx} = 1$

When $y=0$ $\dfrac{dy}{dx} = -1$

When $y = \frac{1}{2}x$, $\dfrac{dy}{dx} = -\dfrac{1}{3}$

When $y = x$, $\dfrac{dy}{dx} = 0$

When $y = 2x$, $\dfrac{dy}{dx} = +\dfrac{1}{3}$

(b) Equation is homogeneous $\dfrac{dy}{dx} = \dfrac{y/x - 1}{y/x + 1}$

let $v = y/x$ $\Rightarrow$ $v + x\dfrac{dv}{dx} = \dfrac{v-1}{v+1}$

$$x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{-v^2-1}{v+1}$$

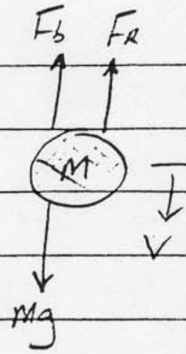
$$\frac{v+1}{v^2+1}\,dv = -\frac{dx}{x}$$

$$\int \frac{v}{v^2+1}\,dv + \int \frac{1}{v^2+1}\,dv = -\ln|x| + C_1$$

$$\frac{1}{2}\ln|v^2+1| + \operatorname{Tan}^{-1}(v) = -\ln|x| + C_1$$

$$\boxed{\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \operatorname{Tan}^{-1}\left(\frac{y}{x}\right) = -\ln|x| + C_1}$$

#7



$$m\frac{dV}{dt} = mg - F_b - F_R$$

$$1000\frac{dV}{dt} = (9800) - (1800) - 200V$$

$$\frac{dV}{dt} = 8 - 0.2V$$

$$\frac{dV}{dt} + 0.2V = 8$$

let $\frac{dV}{dt} = 0 \Rightarrow V_f = \frac{8}{0.2} = 40$ m/s

(a) let $\mu(t) = e^{0.2t}$ then

$$\frac{d}{dt}\left(e^{0.2t}V\right) = 8e^{0.2t}$$

$$e^{0.2t}V = 40e^{0.2t} + C_1$$

$$V = 40 + C_1 e^{-0.2t}$$

$V(0) = 0 \Rightarrow C_1 = -40$

$$\boxed{V(t) = 40\left(1 - e^{-0.2t}\right)}$$

After 5 seconds

$$V(5) = 40\left(1 - e^{-1}\right) = 25.3 \text{ m/s}$$

$$Z(t) = \int_0^t V(t) = \int_0^t \left(40 - 40e^{-0.2t}\right) dt$$

$$= 40t + 200e^{-0.2t} \Big|_0^t$$

$$= 40t + 200\left(e^{-0.2t} - 1\right)$$

$$Z(5) = 200 + 200\,(0.368 - 1)$$

$$= 73.6 \text{ m}$$

$\therefore$ After 5 seconds Velocity $= 25.3$ m/s

Depth $= 73.6$ m

(b) $\lim_{t \to \infty} V(t) = 40$ m/s (Limiting Velocity)

#8 $$R\frac{dQ}{dt} + \frac{Q}{C} = V(t)$$

$$\Rightarrow 20\frac{dQ}{dt} + 100Q = 50e^{-3t}$$

$$\frac{dQ}{dt} + 5Q = 2.5e^{-3t}$$

(a) Time constant $\Rightarrow \frac{1}{5}$ sec. $= RC$

(b) Integration Factor $\mu(t) = e^{5t}$

$$\frac{d}{dt}\left(e^{5t}Q\right) = 2.5e^{2t}$$

$$e^{5t}Q = 1.25e^{2t} + C_1$$

$$Q(t) = 1.25e^{-3t} + C_1e^{-5t}$$

$$Q(0) = 0 \Rightarrow 1.25 + C_1 = 0$$

$$\therefore \boxed{Q(t) = 1.25\left(e^{-3t} - e^{-5t}\right)}$$

To find Q_{max} let $\frac{dQ}{dt} = 0$

$$\Rightarrow 1.25\left(-3e^{-3t} + 5e^{-5t}\right) = 0$$

11a)

$$e^{2t} = \frac{5}{3}$$

$$t_{max} = \frac{1}{2}\ln\left(\frac{5}{3}\right) \approx 0.255 \text{ seconds}$$

$$Q(t_{max}) = 1.25\left(e^{-0.765} - e^{-1.275}\right)$$

$$Q_{max} = 0.232 \text{ Coulombs}$$

Sketch $Q(0) = 0$

$Q_{max} = 0.232$ @ $t = 0.255$

$$\lim_{t \to \infty} Q(t) = 0$$