ME203 – Ordinary Differential Equations
Spring 2000 Midterm Examination

Professor Rob Macdonald **15 June, 2000**

Instructions:

1. Permitted aids: non-programmable scientific calculator and a hand-written equation list consisting of one side of 8½"x11" paper. Integral tables are appended to this exam.
2. Answer **any 5 of the first 6** questions. If you solve all six, the best five will count. Questions **7 and 8 are compulsory**.
3. Do not spend more than 15-20 minutes on any one question!
4. Clear, systematic solutions are required. Part marks will be rewarded for part answers, provided that I can follow your methodology.
5. The time limit is 2 hours.
6. Each question is worth 10 marks (total 70 marks for 7 questions).

Answer any 5 of the first 6 questions:

1. The equation describing the behaviour of a spring-mass-damper system is:

$$m\ddot{x} + c\dot{x} + kx = 0,$$

where *m* is the mass, *c* is the damping and *k* is the spring stiffness. If the system is displaced from equilibrium so that x(0) = 1 m, and is then released from rest, find the equation of motion $x(t)$, if m = 1 kg, k = 2 N/m and the damping parameter is:

(a) c = 3 N-s/m.
(b) c = 2 N-s/m.

2. Find the general solution to the equation:

$$y' + y\tan x = 1,$$

given y(0) = 1.

3. Find the general solution to the first order equation:

$$y' - 2xy = 1,$$

Show that the unique solution for initial condition $y(x_0) = 0$ is:

$$y(x) = e^{x^2}\int_{x_0}^{x} e^{-u^2}\,du$$

4. The equation $\frac{dy}{dx} = \frac{x}{x^2 y + y^3}$ is not separable. Write it in the form of a total differential.
   Show that it is not exact, but that it is possible to find a suitable integrating factor $\mu(y)$.
   Solve the resulting equation given that the curve passes through the point (1,1).

5. The equation $\frac{dy}{dx} = \frac{4y^2 - x^4}{4xy}$ is not separable, but can be made separable by letting $y(x) = xv(x)$. Find the general solution to this equation. Show that an infinite number of solutions can be found which pass through the point (x, y) = (0, 0). Why is this not a violation of the uniqueness theorem?

6. Consider the equation $y' = \frac{y - x}{y + x}$. Sketch the direction field of this equation in the first quadrant ($x \geq 0$, $y \geq 0$) by considering points along the x and y-axes and along the lines
   $y = \frac{1}{2} x$, $y = x$ and $y = 2x$. Find the general solution to the equation.

The following questions are compulsory:

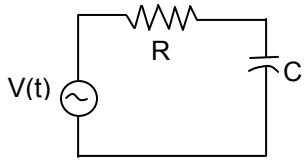
7. A 1000 kg depth charge is dropped in water, starting from rest. Two forces act on it – a buoyancy force of 1800 N and a water resistance force which is 200×V (Newtons), where V is the velocity in m/s. Find the following:

   (a) The velocity and the distance travelled after 5 sec.
   (b) The limiting velocity.

   Assume that $g = 9.8$ m/s^2.

8. The resistor-capacitor circuit shown in the sketch below has the following parameters:

   R = 20 ohms, C = 0.01 farad.



   If the voltage acting on the circuit is $V(t) = 50e^{-3t}$, find:

   (a) The time constant of the circuit
   (b) The maximum charge Q on the capacitor if Q(0) = 0
   (c) Sketch the behaviour of the charge Q(t)