Original Copy

ME203 – Ordinary Differential Equations
S2002 Final Examination

Professor Rob Macdonald **29 July 2002**

Instructions:

(i) This is a **closed-book** exam, but the following aids are permitted:

a) Non-programmable scientific calculator
b) One sheet (8½"x11") hand-written equation list – both sides
c) Tables of integrals and Laplace transforms (these are appended to the back of the examination).

(ii) Answer all 9 questions. Marks are allocated as indicated to the right of each problem (Total 100).

(iii) Clear, systematic solutions are required. Part marks will be rewarded for incomplete answers, provided that I can follow your methodology.

(iv) The time limit is 3 hours.

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15 marks

1. (a) Find the interval of convergence of the following infinite series:

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{n(n-1)2^{2n}}.$$

(b) Find an expression for the following function in terms of Heaviside step functions, then find its Laplace transform:

$$\begin{aligned} f(t) &= t, & 0 \le t \le 1 \\ &= 2-t, & 1 < t \le 2 \\ &= 0, & t > 2 \end{aligned}$$

(c) Find $f(t) = \mathcal{L}^{-1}\left\{ \dfrac{3s^2 + 7s + 6}{(s+1)(s^2+2s+2)} \right\}.$

10 marks

2. Find the general solution to the following problem using the Method of Undetermined Coefficients:

$$2\ddot{y} + 4\dot{y} + 2y = e^{-t} + \sin t$$

10 marks

3. Consider the following differential equation with non-constant coefficients:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - y = 0, \quad x > 0$$

(a) Show that the substitution of independent variable, $x = e^t$, transforms the equation into a linear equation with constant coefficients.

(b) Find the characteristic equation for the resulting transformed ODE, by assuming solutions of the form $y = e^{rt}$

(c) Using the result of part (b), write down the general solution to the equation in terms of the original variable x.

10 marks

4. Suppose we are given the following differential equation:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0$$

(a) What are the singular points of this equation?

(b) Show that one solution to the equation is $y = x^{-1/2}\sin x$.

(c) Find a second, linearly independent solution to this equation using the method of Reduction of Order.

10 marks

5. Consider the following non-homogeneous differential equation:

$$x^2 y'' - 2y = x^2$$

(a) Show that the homogeneous equation $x^2 y'' - 2y = 0$ has two solutions of the form $y = x^r$. Find them.

(b) Verify that the solutions in part (a) are linearly independent on any interval.

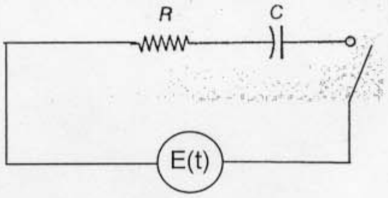
(c) Use the results from part (a) to find the particular solution for the non-homogeneous differential equation.

13 marks

6. The electrical system shown below is described by the following equation:

$$R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)$$

where R is the resistance (Ohms), C is the capacitance (Farads), and E(t) is the applied emf (Volts). For this problem R= 20 kΩ and and C = 0.5 μF (1 μF $=10^{-6}$ F).



(a) The current I(t) in the circuit can be found from $I = dQ/dt$. Show that if you differentiate the above differential equation, you get a first order equation for I(t).

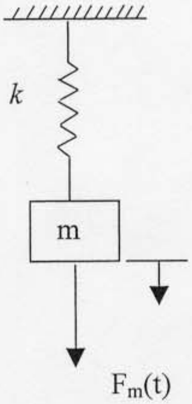
(b) When the switch is closed at t = 0, the voltage E(t) = 20 sin(ωt) acts on the circuit. Use the Method of Undetermined coefficients to find the *steady state* current I(t) in the circuit.

(c) What is the magnitude of the *steady-state* current and the phase angle between the current and E(t) when ω = 100 s^{-1}?

(d) If the applied voltage is 20 Volts DC, what will be the steady-state current in the circuit?

12 marks

7. The spring-mass system shown below has mass $m = 1kg$ and a spring stiffness $k = 25\pi^2\ N/m$. The spring is initially resting at its equilibrium position (y = 0).



At time t = 0 sec, an electromagnet is turned on which exerts a downward force of $F_m = 1N$ on the mass for 1 second, after which the magnet is turned off.

Problem 7 [continued]

(a) Find an expression for the resulting motion of the mass y(t).

(b) Sketch the response curve y(t) versus t for $0 \le t \le 1$ sec.

(c) Sketch the solution to the problem for $t > 1$ sec.

(d) What is the steady state response ($t \to \infty$)?

10 marks

8. Find the solution to the following pair of coupled differential equations:

$$\dot{x}_1 - 2x_1 + x_2 = t - 1$$
$$\dot{x}_2 - 2x_1 + x_2 = t - 2e^{2t}$$

for initial conditions: $x_1(0) = 2;\ x_2(0) = 1$.

10 marks

9. The following ODE has important applications in mathematical physics. It is known as the Hermite equation:

$$y'' - 2xy' + 2my = 0$$

(a) By assuming a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, find the first 6 non-zero terms in the solution.

(b) Find the recursion relationship between a_{n+2} *and* a_n.

(c) What is the interval of convergence of the series in part (a)?

(d) For what values of m (if any) will the solutions of Hermite's equation be simple polynomials?

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Solutions

#1 (a) Consider $\lim_{n\to\infty} \left|\frac{S_{n+1}}{S_n}\right| = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+1)(n)\,2^{2n+2}}\right|}{\left|\frac{(-1)^n (x-2)^n}{n(n-1)\,2^{2n}}\right|}$ ✓(1) ✓(2)

/5

Total 15

$$= \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right) \left|\frac{x-2}{4}\right| = \left|\frac{x-2}{4}\right|$$

For convergence $\left|\frac{x-2}{4}\right| < 1$

$$\Rightarrow \quad -4 < x-2 < 4 \quad (1)$$
$$-2 < x < 6$$

When $x = -2$ $\quad y(x) = \sum_{n=0}^{\infty} \frac{(-1)^{2n} 4^n}{n(n-1)4^n} = \sum_{n=0}^{\infty} \frac{1}{n(n-1)}$

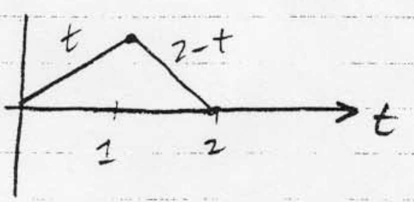
When $x = +6$ $\quad y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{n(n-1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n(n-1)}$

Both series converge by comparison with $\sum \frac{1}{n^2}$

$\therefore$ $x \in [-2, 6]$ is interval of convergence (1)

/5 2

(b)



$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\mathcal{L}\{f\}$

$$f(t) = t\,(1 - H(t-1)) + (2-t)(H(t-1) - H(t-2))$$

$$= t - 2tH(t-1) + 2H(t-1) + (t-2)H(t-2)$$

③ $$= t - 2((t-1)+1)H(t-1) + 2H(t-1) + (t-2)H(t-2)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} - \cancel{\frac{2e^{-s}}{s}} + \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s^2} \quad ①$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2}(1 - 2e^{-s} + e^{-2s}) = \frac{1}{s^2}(1-e^{-s})^2 \checkmark$$

(c) Let $\dfrac{3s^2+7s+6}{(s+1)(s^2+2s+2)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+2s+2}$ /5

$$\Rightarrow As^2 + 2As + 2A + Bs^2 + Cs + Bs + C = 3s^2 + 7s$$ ②

$\therefore$ $(A+B) = 3$ (i)

$(2A+B+C) = 7$ (ii)

$2A + C = 6$ (iii)

(ii) − (iii) $\Rightarrow$ $B = 1$

$\therefore A = 2$

$C = 2$ ✓ ①

$$\mathcal{L}^{-1}\left\{ \frac{2}{s+1} + \frac{s+2}{(s+1)^2+1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{2}{s+1} \right\} + \mathcal{L}^{-1}\left\{ \frac{(s+1)}{(s+1)^2+1} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2+1} \right\}$$

$$= ② \; 2e^{-t} + e^{-t}\cos t + e^{-t}\sin t \quad /5$$

#2 $\quad 2\ddot{y} + 4\dot{y} + 2y = e^{-t} + \sin t$

Homogeneous equation $\quad \ddot{y}_c + 2\dot{y}_c + y_c = 0$

let $y_c = e^{rt}$ ② $\qquad r^2 + 2r + 1 = 0 \; ; \; r = -1$ (repeated root ①

$$y_c = c_1 e^{-t} + c_2 t e^{-t} \quad ②$$

For a particular solution we would normally try:

$$y_p = At^2 e^{-t} + B\sin t + C\cos t$$

However since e^{-t} and te^{-t} are complementary solutions, try

$$y_p = At^2 e^{-t} + B\sin t + C\cos t$$

$$y_p' = 2Ate^{-t} - At^2e^{-t} + B\cos t - C\sin t$$

$$② \; y_p'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t} - B\sin t - C\cos t$$

$$2y_p'' + 4y_p' + 2y_p$$
$$= At^2e^{-t}(2-4+2) + Ate^{-t}(8-8)$$
$$+ 4Ae^{-t} + \sin t\,(2B-4C-2B)$$
$$+ \cos t\,(2C+4B-2C)$$
$$= 4Ae^{-t} - 4C\sin t + 4B\cos t = e^{-t} + \sin t$$

$$\therefore \quad A = 1/4 \quad C = -1/4 \quad B = 0$$

$$y_p(t) = \frac{1}{4}t^2e^{-t} - \frac{1}{4}\cos t \quad 2$$

General Solution $y(t) = y_c(t) + y_p(t)$

$$= c_1e^{-t} + c_2te^{-t} + \frac{1}{4}t^2e^{-t} - \frac{1}{4}\cos t \quad 1$$

/10

$/10$ 5

Total

#3 let $x = e^t \quad t = \ln x$

(a) $\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{d}{dt}$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{d}{dt}\right) = -\frac{1}{x^2}\frac{d}{dt} + \frac{1}{x^2}\frac{d^2}{dt^2}$$

$\therefore \quad x^2y'' + xy' - y$

$$= -\frac{dy}{dt} + \frac{d^2y}{dt^2} + \frac{dy}{dt} - y$$

$/6$

$$= \frac{d^2y}{dt^2} - y \quad = 0$$

(b) Let $y = e^{rt}$

$r^2 - 1 = 0 ; \; r = \pm 1$ (repeated root) $/2$

$$y = c_1 e^t + c_2 e^{-t}$$

(c) $y(x) = c_1 x + c_2 x^{-1}$ $/2$

#4 $y'' + \frac{1}{x}y' + \left(\frac{4x^2-1}{4x^2}\right)y = 0$

/10

(a) One singular point $x=0$

where $p(x) = \frac{1}{x}$ and $q(x) = \frac{4x^2-1}{4x^2}$ are undefined

2

(b) let $y_1 = x^{-1/2}\sin x$

$y_1' = x^{-1/2}\cos x - \frac{1}{2}x^{-3/2}\sin x$

$y_1'' = -x^{-1/2}\sin x - x^{-3/2}\cos x + \frac{3}{4}x^{-5/2}\sin x$

$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y$

$= \cancel{-x^{-1/2}\sin x} - \cancel{x^{-3/2}\cos x} + \cancel{\frac{3}{4}x^{-5/2}\sin x}$

$+ \cancel{x^{-3/2}\cos x} - \cancel{\frac{1}{2}x^{-5/2}\sin x}$

$+ \cancel{x^{-1/2}\sin x} - \cancel{\frac{1}{4}x^{-5/2}\sin x}$

$= 0$

3

(c) Let $y_2 = v(x)y_1(x)$

$y_2' = v'y_1 + y_1'v$

$y_2'' = v''y_1 + 2v'y_1' + vy_1''$

$\therefore y_2'' + \frac{1}{x}y_2' + \left(1 - \frac{1}{4x^2}\right)y_2$

$= v(x)\left[\cancel{y_1'' + \frac{1}{x}y_1' + \left(1-\frac{1}{4x^2}\right)y_1}\right]^{\to 0}$

$+ \frac{1}{x}v'y_1 + v''y_1 + 2v'y_1' = 0$

$$\therefore \quad y_1 v'' + \left(2y_1' + \frac{y_1}{x}\right) v' = 0$$

$y_1 = x^{-1/2} \sin x$

$y_1' = -\frac{1}{2} x^{-3/2} \sin x + x^{-1/2} \cos x$

$$x^{-1/2} \sin x \, v'' + \left(\cancel{-x^{-3/2} \sin x} + 2x^{-1/2} \cos x + \cancel{x^{-3/2} \sin x}\right) = 0$$

$$v'' + \frac{2 \cos x}{\sin x} v' = 0 \qquad v'' + 2\cot x \, v' = 0$$

let $u = v'$

$$\frac{u'}{u} = -2 \frac{\cos x}{\sin x}$$

$$\ln u = -2 \ln(\sin x)$$

$$u = \frac{1}{\sin^2 x} = \csc^2 x$$

$$v = \int u \, dx = \int \csc^2 x \, dx = \cot x$$

$$\therefore y_2(x) = v(x) y_1(x)$$

$$= \frac{\cos x}{\sin x} x^{-1/2} \sin x$$

$$= x^{-1/2} \cos x \quad 5$$

#5 $x^2y'' - 2y = x^2$

$\frac{}{10}$

(a) $x^2y'' - 2y = 0$

let $y = x^r$

$$x^2(r(r-1))x^{r-2} - 2x^r = 0$$

$$(r^2 - r - 2)\, x^r = 0$$

$$(r-2)(r+1)\, x^r = 0$$

$$r = 2; \quad r = -1$$

$$\therefore \quad y_H(x) = c_1 x^2 + c_2 x^{-1} \quad 3$$

(b) $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & \frac{1}{x} \\ 2x & -\frac{1}{x^2} \end{vmatrix}$

$$= -1 - 2 = -3$$

3

Since $W(y_1, y_2) \neq 0$ anywhere, they are linearly independent on any interval $x \in (-\infty, \infty)$

(c) We try solution of the form

$$y_p = u_1 y_1 + u_2 y_2$$

where $y_1 = x^2$

$y_2 = 1/x$ 4

Two equations to solve for Variation of Parameters method

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_1' = f(x) = 1$$

Note $f(x) \neq x^2$ in Standard form.

$$\therefore \quad \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_1' = -\frac{y_2}{W(y_1, y_2)} = \frac{1}{3x}$$

$$u_2' = \frac{y_1}{W(y_1, y_2)} = -\frac{x^2}{3}$$

$$u_1 = \int \frac{dx}{3x} = \frac{1}{3} \ln x$$

$$u_2 = \int -\frac{x^2}{3} dx = -\frac{x^3}{9}$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2$$

$$y_p = \frac{1}{3} x^2 \ln x - \frac{x^2}{9}$$

#6 R-C circuit

/13

(a) $$R\dot{Q} + \frac{Q}{C} = E(t)$$

$$\dot{Q} + \frac{Q}{RC} = \frac{E(t)}{R} \quad (*)$$

Let $I = \dot{Q}$

Take derivatives of (*)

$$\boxed{\frac{dI}{dt} + \frac{1}{RC} I = \frac{1}{R}\frac{dE}{dt}}$$ 3

(b) For $R = 20\times10^{3}\,\Omega$ $C = 0.5\times10^{-6}\,F$ $V(t) = 20\sin\omega t$

$$RC = 0.01\,s \qquad \frac{dV}{dt} = 20\omega\cos\omega t$$

$$\therefore \frac{dI}{dt} + 100I = 10^{-3}\omega\cos\omega t$$

let $I = A\cos\omega t + B\sin\omega t$

$$\frac{dI}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$100A + B\omega = 10^{-3}\omega$$

$$-A\omega + 100B = 0$$

$$A = \frac{\begin{vmatrix} 10^{-3}\omega & \omega \\ 0 & 100 \end{vmatrix}}{\begin{vmatrix} 100 & \omega \\ -\omega & 100 \end{vmatrix}} = \frac{0.1\omega}{10^4+\omega^2}$$

$$B = \frac{\begin{vmatrix} 100 & 10^{-3}\omega \\ -\omega & 0 \end{vmatrix}}{\omega^2+10^4} = \frac{10^{-3}\omega^2}{10^4+\omega^2}$$

$$\therefore I(t) = \frac{0.1\omega}{10^4+\omega^2}\cos\omega t + \frac{10^{-3}\omega^2}{10^4+\omega^2}\sin\omega t \quad 5$$

(c) If $\omega = 10^2$

$$I(t) = \frac{10}{2\times10^4}\cos\omega t + \frac{10}{2\times10^4}\sin\omega t$$

$$|I_{max}| = \left[\left(\frac{10}{2\times10^4}\right)^2 + \left(\frac{10}{2\times10^4}\right)^2\right]^{1/2}$$

$$= \left[\sqrt{2}\right]\frac{10}{2\times10^4} \quad 3$$

$$= \frac{\sqrt{2}}{2}\times10^{-3} = 7.07\times10^{-4} = \underline{0.000707}$$

$$\tan\varphi = \frac{10/2\times10^4}{10/2\times10^4} = 1 \Rightarrow \varphi = 45^\circ \quad //$$

(leading)

(d) When $E=20$, $\frac{dV}{dt}=0$

2

so we get $\frac{dI}{dt} + \frac{1}{RC}I = 0 \Longrightarrow I_{ss} = 0$

12

#7 $m\ddot{y} + ky = F_m(t)$

$$\ddot{y} + 25\pi^2 y = 1 - H(t-1)$$

$$s^2 \mathcal{L}\{y\} + 25\pi^2 \mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{H(t-1)\}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+25\pi^2}\left[\frac{1}{s} - \frac{e^{-as}}{s}\right]$$

let $\frac{1}{s(s^2+25\pi^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+25\pi^2}$

$$\Longrightarrow \quad As^2 + 25\pi^2 A + Bs^2 + Cs = 1$$

$$A+B=0$$
$$C=0$$
$$A = 1/25\pi^2 \qquad B = -1/25\pi^2$$

$$\therefore \mathcal{L}\{y\} = \frac{1}{25\pi^2}\left(\frac{1}{s} - \frac{s}{s^2+25\pi^2}\right)\left(1-e^{-as}\right)$$ 4

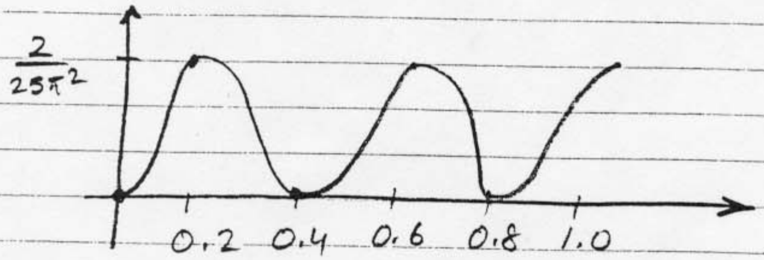
$$= \frac{1}{25\pi^2}\left(\mathcal{L}\{1\} - \mathcal{L}\{\cos 5\pi t\}\right)\left(1-e^{-as}\right)$$

Since $\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

$$\Longrightarrow y(t) = \frac{1}{25\pi^2}\left(1 - \cos 5\pi t\right)$$
$$- \frac{1}{25\pi^2}\left(H(t-1) - H(t-1)\cos 5\pi(t-1)\right)$$

(b) For $0 \le t \le 1$ $\quad H(t-1) = 0$

$$\therefore\ y(t) = \frac{1}{25\pi^2}\left(1 - \cos 5\pi t\right)$$



(2.5)

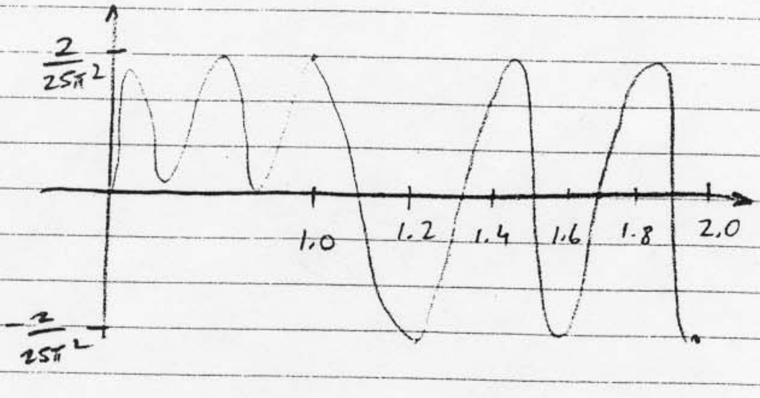
(c) For $t > 1$, Solution is

$$\frac{1}{25\pi^2}\left(\cos 5\pi(t-1) - \cos 5\pi t\right)$$

$$= \frac{1}{25\pi^2}\left(\cos 5\pi t \overset{-1}{\cos 5\pi} + \sin 5\pi t \,\overset{0}{\cancel{\sin 5\pi}} - \cos 5\pi t\right)$$

$$= \frac{-2}{25\pi^2}\left(\cos 5\pi t\right)$$

(2.5)



For $t \to \infty$ $\quad y(t) = -\frac{2}{25\pi^2} \cos 5\pi t$

#8 $\dot{x}_1 - 2x_1 + x_2 = t-1$ $x_1(0) = 2$

$\dot{x}_2 - 2x_1 + x_2 = t - 2e^{2t}$ $x_2(0) = 1$

$\frac{1}{10}$

$(D-2)x_1 + x_2 = t-1$ (1)

$-2x_1 + (D+1)x_2 = t - 2e^{2t}$ (2)

Multiply (1) by $(D+1)$ and subtract (2) $\times$ 1

$$(D-2)(D+1)\,x_1 + 2x_1 = (D+1)(t-1) - t + 2e^{2t}$$

$$(D^2 - D - 2)x_1 + 2x_1 = 1 + t - 1 - t + 2e^{2t}$$

$$\ddot{x}_1 - \dot{x}_1 = 2e^{2t}$$

$$x_2'' - x_2' = -1$$

Characteristic Equation $r^2 - r = 0 \Rightarrow r = 0, 1$

$$x_{1c} = c_1 + c_2 e^t$$

For particular solution try

$$x_{1c} = Ae^{2t}$$

$$\Rightarrow 4Ae^{2t} - 2Ae^{2t} = 2e^{2t}$$

$$A = 1$$

$$\therefore \boxed{x_1(t) = c_1 + c_2 e^t + e^{2t}}$$ 4

From 1st ODE: $x_2(t) = t - 1 + 2x_1 - \dot{x}_1$

$$= (t-1) + 2c_1 + 2c_2 e^t + 2e^{2t} - c_2 e^t - 2e$$

$$x_2 = t - 1 + 2c_1 + c_2 e^t$$

$$\boxed{X_2 = t + 2C_1 - 1 + C_2 e^t}$$ 3

$X_1(0) = 2 \implies C_1 + C_2 + 1 = 2$

$X_2(0) = 1 \longrightarrow 2C_1 - 1 + C_2 = 1$

$$\begin{cases} C_1 + C_2 = 1 \\ 2C_1 + C_2 = 2 \end{cases} \implies \begin{matrix} C_1 = 1 \\ C_2 = 0 \end{matrix}$$

$$\therefore \begin{cases} X_1(t) = 1 + e^{2t} \\ X_2(t) = t + 1 \end{cases}$$ B

#9 $y'' - 2xy' + 2my = 0$ $\frac{}{10}$

(a) $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$; $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^n + 2m \sum_{n=0}^{\infty} a_n x^n = 0$

$\downarrow$

$n \to n+2$

$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n=1}^{\infty} n a_n x^n + 2m \sum_{n=0}^{\infty} a_n x^n = 0$

For $n=0$ $\quad 2 \cdot 1\, a_2 + 2m a_0 = 0 \quad a_2 = -m a_0 = -2$

For $n=1$ $\quad 3 \cdot 2 a_3 - 2 a_1 + 2m a_1 = 0 \quad a_3 = \frac{-2(m-1) a_1}{3 \cdot 2}$

For $n=2$ $\quad 4 \cdot 3 a_4 - 2 \cdot 2 a_2 + 2m a_2 = 0$

$a_4 = \frac{-2(m-2) a_2}{4 \cdot 3}$

$= \frac{2^2 (m-2) m a_0}{4!}$

For $n=3$ $\quad 5 \cdot 4 a_5 - 2 \cdot 3 a_2 + 2m a_2 = 0$

$a_5 = \frac{-2(m-3) a_3}{5 \cdot 4}$

$= \frac{2^2 (m-3)(m-1) a_3}{5!}$

$\therefore y(x) = a_0 \left(1 - m x^2 + \frac{1}{6} (m)(m-2) x^4 + \cdots \right)$ 3

$+ a_1 \left(x - \frac{m-1}{3} x^3 + \frac{(m-3)(m-1)}{30} x^5 + \cdots \right)$

(b) $(n+2)(n+1)a_{n+2} - 2na_n + 2m a_n = 0$

$$a_{n+2} = \frac{-2(m-n)}{(n+2)(n+1)} a_n \qquad (2)$$

Even $\quad a_{2n} = \dfrac{(-1)^n (m-(2n-2))(m-(2n-4))\cdots m\, a_0}{(2n)!}$ 2

Odd $\quad a_{2n+1} = (-1)^n \dfrac{(m-(2n-1))(m-(2n-3))\cdots(m-1)a_1}{(2n+1)!}$ 2

(c) There are no singularities in the equation

$\therefore$ series converges on $(-\infty, \infty)$ (1)

(d) If m is even, then even series terminates

at x^m (1)

Similarly if m is odd, the odd series terminates

at x^m