ME203 PROBLEM SET #8

1. Text – Section 5.3

31.

Solution:

Solving this problem, we follow the arguments described in Section 5.1, page 261 of the text, i.e., $x(t)$, the mass of salt in the tank A, and $y(t)$, the mass of salt in the tank B, satisfy the system

$$\frac{dx}{dt} = \text{input}_A - \text{output}_A \qquad (1)$$
$$\frac{dy}{dt} = \text{input}_B - \text{output}_B$$

with initial conditions $x(0) = 0$, $y(0) = 200$. It is important to notice that the volume of each tank stays at 100 L because the net flow rate into each tank is the same as the net outflow. Next we observe that "input_A" consists of the salt coming from outside, which is

$$2\,\text{kg/L} \cdot 6\,\text{L/min} = 12\,\text{kg/min}$$

and the salt coming from the tank B, which is given by

$$\frac{y(t)}{100}\text{kg/L} \cdot 1\text{L/min} = \frac{y(t)}{100}\text{kg/min}$$

Thus,

$$\text{input}_A = \left(12 + \frac{y(t)}{100}\right)\text{kg/min}$$

"output_A" consists of two flows: one is going out of the system and the other one is going to the tank B. So,

$$\text{output}_A = \frac{x(t)}{100}\text{kg/L} \cdot (4+3)\,\text{L/min}$$
$$= \frac{7x(t)}{100}\text{kg/min}$$

and the first equation in (1) becomes

$$\frac{dx}{dt} = 12 + \frac{y}{100} - \frac{7x}{100}$$

Similarly, the second equation in (1) can be written as

$$\frac{dy}{dt} = \frac{3x}{100} - \frac{3y}{100}$$

Rewriting this system in the operator form, we obtain

$$(D+0.07)[x] - 0.01y = 12,$$
$$-0.03x + (D+0.03)[y] = 0 \qquad (2)$$

Eliminating y yields

$$\{(D+0.07)(D+0.03) - (-0.01)(-0.03)\}[x]$$
$$= (D+0.03)[12] = 0.36$$

which simplifies to

$$\left(D^2 + 0.1D + 0.0018\right)[x] = 0.36 \qquad (3)$$

The auxiliary equation,

$$r^2 + 0.1r + 0.0018 = 0$$

has roots

$$r_1 = \frac{-5-\sqrt{7}}{100} \approx -0.0765$$
$$r_2 = \frac{-5+\sqrt{7}}{100} \approx -0.0235$$

Therefore, the general solution to the corresponding homogeneous equation is

$$x_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Since the non-homogeneous term in (3) is a constant (0.36), we are looking for a particular solution of the form

$$x_p(t) = A = \text{const}.$$

Substituting into (3) yields

$$0.0018A = 0.36 \qquad \Rightarrow A = 200$$

and the general solution, $x(t)$, is

$$x(t) = x_h(t) + x_p(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + 200$$

From the first equation in (2) we find

$$y(t) = 100 \cdot ((D + 0.07I)[x] - 12)$$
$$= 100\frac{dx}{dt} + 7x(t) - 1200$$
$$= 100\{C_1 e^{r_1 t} + C_2 e^{r_2 t}\}$$
$$+ 7\{C_1 e^{r_1 t} + C_2 e^{r_2 t} + 200\} - 1200$$
$$= \left(2-\sqrt{7}\right)C_1 e^{r_1 t} + \left(2+\sqrt{7}\right)C_2 e^{r_2 t} + 200$$

The initial conditions imply

$$0 = x(0) = C_1 + C_2 + 200$$
$$200 = y(0) = \left(2-\sqrt{7}\right)C_1 + \left(2+\sqrt{7}\right)C_2 + 200$$
$$\Rightarrow \begin{array}{l} C_1 + C_2 = -200 \\ \left(2-\sqrt{7}\right)C_1 + \left(2+\sqrt{7}\right)C_2 = 0 \end{array}$$

$$\Rightarrow C_1 = -\left(100 + \frac{200}{\sqrt{7}}\right) \quad C_1 = -\left(100 - \frac{200}{\sqrt{7}}\right)$$

Thus the solution to the problem is

$$x(t) = -\left(100 + \frac{200}{\sqrt{7}}\right)e^{r_1 t} - \left(100 - \frac{200}{\sqrt{7}}\right)e^{r_2 t} + 200\text{kg}$$

$$y(t) = \frac{300}{\sqrt{7}}e^{r_1 t} - \frac{300}{\sqrt{7}}e^{r_2 t} + 200\text{kg}$$

2. Text – Section 7.2

11. $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t \end{cases}$

Solution:

As in Example 4 on page 375 of the text, we first break the integral into separate parts. Thus,

$$\int_0^\infty e^{-st} f(t)dt = \int_0^\pi e^{-st}\sin t dt + \int_\pi^\infty e^{-st}\cdot 0 dt$$
$$= \int_0^\pi e^{-st}\sin t dt$$

Referring to the table of integrals on the inside front cover of the text, we see that,

$$\int_0^\infty e^{-st} f(t)dt = \int_0^\pi e^{-st}\sin t dt$$
$$= \left.\frac{e^{-st}(-s\sin t - \cos t)}{s^2+1}\right|_0^\pi$$
$$= \frac{e^{-\pi s}+1}{s^2+1} \quad \text{for all } s$$

13. $£\{6e^{-3t} - t^2 + 2t - 8\}$

Solution:

By the linearity of the Laplace transform,

$$£\{6e^{-3t} - t^2 + 2t - 8\}$$
$$= 6£\{e^{-3t}\} - £\{t^2\} + 2£\{t\} - £\{8\}$$

From Table 7.1 on page 380 of the text, we see that

$$£\{e^{-3t}\} = \frac{1}{s+3}, \quad s > -3 \qquad (4)$$

$$£\{t^2\} = \frac{2!}{s^3}, \quad s > 0 \qquad (5)$$

$$£\{t\} = \frac{1!}{s^2}, \quad s > 0 \qquad (6)$$

$$£\{8\} = \frac{8}{s}, \quad s > 0 \qquad (7)$$

Thus,

$$£\{6e^{-3t} - t^2 + 2t - 8\}$$
$$= \frac{6}{s+3} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{8}{s}$$

Since (4), (5), (6) and (7) all hold for $s > 0$, we see that our answer is valid for $s > 0$.

17. $£\{e^{3t}\sin 6t - t^3 + e^t\}$

Solution:

By the linearity of the Laplace transform,

$$£\{e^{3t}\sin 6t - t^3 + e^t\}$$
$$= £\{e^{3t}\sin 6t\} - £\{t^3\} + £\{e^t\}$$

From Table 7.1 on page 380 of the text, we see that

$$£\{e^{3t}\sin 6t\} = \frac{6}{(s-3)^2+36}, \quad s > 3 \qquad (8)$$

$$£\{t^3\} = \frac{3!}{s^4}, \quad s > 0 \qquad (9)$$

$$£\{e^t\} = \frac{1}{s-1}, \quad s > 1 \qquad (10)$$

Thus,

$$£\{e^{3t}\sin 6t - t^3 + e^t\}$$
$$= \frac{6}{(s-3)^2+36} - \frac{6}{s^4} + \frac{1}{s-1}$$

Since (8), (9), and (10) all hold for $s > 3$, we see that our answer is valid for $s > 3$.

20. $£\{e^{-2t}\cos\sqrt{3}t - t^2 e^{-2t}\}$

Solution:

By the linearity of the Laplace transform,

$$£\{e^{-2t}\cos\sqrt{3}t - t^2 e^{-2t}\}$$
$$= £\{e^{-2t}\cos\sqrt{3}t\} - £\{t^2 e^{-2t}\}$$

From Table 7.1 on page 380 of the text, we see that

$$£\{e^{-2t}\cos\sqrt{3}t\} = \frac{s+2}{(s+2)^2+3}, \quad s > -2 \qquad (11)$$

$$\pounds\{t^2 e^{-2t}\} = \frac{2!}{(s+2)^3}, \qquad s > -2 \qquad (12)$$

Thus,

$$\pounds\{e^{-2t}\cos\sqrt{3}t - t^2 e^{-2t}\}$$
$$= \frac{s+2}{(s+2)^2+3} + \frac{2}{(s+2)^3}$$

Since (11) and (12) all hold for $s > -2$, we see that our answer is valid for $s > -2$.

3. Text – Section 7.3

5. $2t^2 e^{-t} - t + \cos 4t$

Solution:

From Table 7.1 on page 380 of the text, we see that

$$\pounds\{t^2 e^{-t}\} = \frac{2!}{(s+1)^3} = \frac{2}{(s+1)^3}, \qquad s > -1$$

$$\pounds\{t\} = \frac{1}{s^2}, \qquad s > 0$$

$$\pounds\{\cos 4t\} = \frac{s}{s^2+16}, \qquad s > 0$$

Thus,

$$\pounds\{2t^2 e^{-t} - t + \cos 4t\}$$
$$= \frac{4}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2+16}, \qquad s > 0$$

7. $(t-1)^4$

Solution:

$$(t-1)^4 = t^4 - 4t^3 + 6t^2 - 4t + 1$$

From Table 7.1 on page 380 of the text, we see that

$$\pounds\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5}, \qquad s > 0$$

$$\pounds\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}, \qquad s > 0$$

$$\pounds\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}, \qquad s > 0$$

$$\pounds\{t\} = \frac{1}{s^2}, \qquad s > 0$$

$$\pounds\{1\} = \frac{1}{s}, \qquad s > 0$$

Thus

$$\pounds\{(t-1)^4\}$$
$$= \frac{24}{s^5} - \frac{24}{s^4} + \frac{12}{s^3} - \frac{4}{s^2} + \frac{1}{s}, \qquad s > 0$$

15. $\cos^3 t$

Solution:

From the trigonometric identity

$$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$$

we find that

$$\cos^3 t = \cos t \cos^2 t = \frac{1}{2}\cos t + \frac{1}{2}\cos t \cos 2t$$

Next we write

$$\cos t \cos 2t = \frac{1}{2}[\cos(2t+t) + \cos(2t-t)]$$
$$= \frac{1}{2}\cos 3t + \frac{1}{2}\cos t$$

Thus

$$\cos^3 t = \frac{1}{2}\cos t + \frac{1}{4}\cos 3t + \frac{1}{4}\cos t$$
$$= \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$$

We now use the linearity of the Laplace transform and Table 7.1 on page 380 of the text to find that

$$\pounds\{\cos^3 t\} = \frac{3}{4}\pounds\{\cos t\} + \frac{1}{4}\pounds\{\cos 3t\}$$
$$= \frac{3s}{4(s^2+1)} + \frac{s}{4(s^2+9)}, \qquad s > 0$$

25. b) $\pounds\{t^2 \cos bt\}$

Solution:

Using formula (6) on page 385

$$\pounds\{t^2 \cos bt\} = \frac{d^2}{ds^2}\pounds\{\cos bt\} = \frac{d^2}{ds^2}\left[\frac{s}{s^2+b^2}\right]$$
$$= \frac{d}{ds}\left[\frac{b^2 - s^2}{(s^2+b^2)^2}\right] = \frac{2s^3 - 6sb^2}{(s^2+b^2)^3}, \qquad s > 0$$