ME203 PROBLEM SET #6

1. Text – Section 4.5

35. $\dfrac{d^2w}{dx^2}+\dfrac{6}{x}\dfrac{dw}{dx}+\dfrac{4}{x^2}w=0$

Solution:

First multiply this equation by x^2 (which we can do since $x>0$) to transform it into the Cauchy-Euler equation given by

$$x^2w''(x)+6xw'(x)+4w(x)=0$$

Then by making the substitution $x=e^t$ (and using equation (17) on page 188 of the text), we transform the Cauchy-Euler equation into an equation with constant coefficients given by

$$\left(\frac{d^2w}{dt^2}-\frac{dw}{dt}\right)+6\frac{dw}{dt}+4w=0$$

or

$$\frac{d^2w}{dt^2}+5\frac{dw}{dt}+4w=0 \qquad (1)$$

This is a linear equation with constant coefficients and has the associated auxiliary equation

$$r^2+5r+4=0$$

which has roots $r=-1,-4$.

Therefore, a general solution to equation (1) is

$$w(t)=c_1e^{-t}+c_2e^{-4t}=c_1\left(e^t\right)^{-1}+c_2\left(e^t\right)^{-4}$$

To change this equation back into one with the independent variable x, we again use the substitution $x=e^t$.

Therefore the solution becomes

$$w(x)==c_1x^{-1}+c_2x^{-4}$$

2. Text – Section 4.6

33. $x^2y''(x)-3xy'(x)+6y(x)=0$

Solution:

Making the substitution $x=e^t$ (and using equation (17) on page 188 of the text), we transform the Cauchy-Euler equation into an equation with constant coefficients given by

$$\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)-3\frac{dy}{dt}+6y=0$$

or

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+6y=0 \qquad (2)$$

This is a linear equation with constant coefficients and has the associated auxiliary equation

$$r^2-4r+6=0$$

which has roots $r=2\pm\sqrt{2}i$.

Therefore, a general solution to equation (2) is

$$y(t)=c_1e^{2t}\cos\left(\sqrt{2}t\right)+c_2e^{2t}\sin\left(\sqrt{2}t\right)$$
$$=c_1\left(e^t\right)^2\cos\left(\sqrt{2}t\right)+c_2\left(e^t\right)^2\sin\left(\sqrt{2}t\right)$$

To change this equation back into one with the independent variable x, we again use the substitution $x=e^t$ or, solving this expression for t, $t=\ln x$.

Therefore the solution becomes

$$y(x)=c_1x^2\cos\left(\sqrt{2}\ln x\right)+c_2x^2\sin\left(\sqrt{2}\ln x\right)$$

3. Text – Section 6.2

3. $6z'''+7z''-z'-2z=0$

Solution:

The auxiliary equation for this problem is

$$6r^3+7r^2-r-2=0$$

By inspection we see that $r=-1$ is a root to this equation and so we can factor it as follows

$$6r^3+7r^2-r-2=(r+1)(6r^2+r-2)$$
$$=(r+1)(3r+2)(2r-1)$$
$$=0$$

Thus, we see that the roots to the auxiliary equation are $r=-1,-\dfrac{2}{3},\dfrac{1}{2}$.

These roots are real and non-repeating. Therefore, a general solution to this problem is give by

$$z(x)=c_1e^{-x}+c_2e^{-2x/3}+c_3e^{x/2}$$

13. $y^{(4)}+4y''+4y=0$

Solution:

The auxiliary equation for this problem is

$$r^4+4r^2+4=0$$

This can be factored as

$$\left(r^2+2\right)=0$$

Therefore, this equation has roots

$$r = \sqrt{2}i, -\sqrt{2}i, \sqrt{2}i, -\sqrt{2}i\ ,$$

which we see are repeated and complex. Therefore, a general solution to this problem is give by

$$y(x) = c_1 \cos\left(\sqrt{2}x\right) + c_2 x \cos\left(\sqrt{2}x\right) + c_3 \sin\left(\sqrt{2}x\right) + c_4 x \sin\left(\sqrt{2}x\right)$$

4. Text – Section 4.8

6. $y'' - y' - 2y = -2x^3 - 3x^2 + 8x + 1$

Solution:

According to Table 4.1 on page 208 of the text, this non-homogeneous term is of Type I. Thus, we want a particular solution of this differential equation to have the form

$$y_p(x) = A_3x^3 + A_2x^2 + A_1x + A_0\ .$$

Therefore, y_p, y_p' and y_p'' are given by

$$y_p(x) = A_3x^3 + A_2x^2 + A_1x + A_0$$

$$y_p'(x) = 3A_3x^2 + 2A_2x + A_1\text{, and}$$

$$y_p''(x) = 6A_3x + 2A_2$$

Substituting these expressions into the original differential equation yields

$$\begin{aligned} & y_p'' - y_p' - 2y_p \\ &= 6A_3x + 2A_2 - \left(3A_3x^2 + 2A_2x + A_1\right) \\ &\qquad - 2\left(A_3x^3 + A_2x^2 + A_1x + A_0\right) \\ &= -2A_3x^3 - \left(3A_3 + 2A_2\right)x^2 \\ &\quad + \left(6A_3 - 2A_2 - 2A_1\right)x + \left(2A_2 - A_1 - 2A_0\right) \\ &= -2x^3 - 3x^2 + 8x + 1 \end{aligned}$$

By equating coefficients, we obtain

$$\begin{aligned} -2A_3 &= -2 && \Rightarrow A_3 = 1 \\ 3A_3 + 2A_2 &= 3 && \Rightarrow A_2 = 0 \\ 6A_3 - 2A_2 - 2A_1 &= 8 && \Rightarrow A_1 = -1 \\ 2A_2 - A_1 - 2A_0 &= 1 && \Rightarrow A_0 = 0 \end{aligned}$$

Therefore, a particular solution of the non-homogeneous differential equation

$$y_p'' - y_p' - 2y_p = -2x^3 - 3x^2 + 8x + 1$$

is given by

$$y_p(x) = x^3 - x$$

7. $y'' - y' + 9y = 3\sin 3x$

Solution:

According to Table 4.1 on page 208 of the text, this non-homogeneous term is of Type III (with $a = 0$, $b = 3$, and $\beta = 3$). Thus, we want a particular solution of this differential equation to have the form

$$y_p(x) = A\cos 3x + B\sin 3x\ .$$

Therefore, y_p, y_p' and y_p'' are given by

$$y_p(x) = A\cos 3x + B\sin 3x$$

$$y_p'(x) = -3A\sin 3x + 3B\cos 3x\text{, and}$$

$$y_p''(x) = -9A\cos 3x - 9B\sin 3x$$

Substituting these expressions into the original differential equation yields

$$\begin{aligned} & y_p'' - y_p' + 9y_p \\ &= -9A\cos 3x - 9B\sin 3x + \left(3A\sin 3x - 3B\cos 3x\right) \\ &\qquad + 9\left(A\cos 3x + B\sin 3x\right) \\ &= 3A\sin 3x - 3B\cos 3x \\ &= 3\sin 3x \end{aligned}$$

By equating coefficients, we obtain

$$\begin{aligned} 3A &= 3 && \Rightarrow A = 1 \\ -3B &= 0 && \Rightarrow B = 0 \end{aligned}$$

Therefore, a particular solution of the non-homogeneous differential equation

$$y_p'' - y_p' + 9y_p = 3\sin 3x$$

is given by

$$y_p(x) = \cos 3x$$

9. $\dfrac{d^2\theta}{dr^2} - 5\dfrac{d\theta}{dr} + 6\theta = re^r$

Solution:

For this problem, the corresponding homogeneous equation is

$$\theta'' - 5\theta' + 6\theta = 0$$

which has the associated auxiliary equation

$$\rho^2 - 5\rho + 6 = 0$$

This auxiliary equation has roots $\rho = 2,3$.

Thus, a general solution of this homogeneous equation is given by

$$\theta_h(r) = C_1e^{2r} + C_2e^{3r}$$

The non-homogenous term of the original differential equation is re^r. According to Table 4.1 on page 208 of the text, this non-homogeneous term is of Type IV. Thus, we want a particular solution of this differential equation to have the form $\theta_p(r) = r^s(A_1 r + A_0)e^r$.

Since neither re^r nor e^r are solutions of the corresponding homogeneous equation, we let $s = 0$.

Therefore, θ_p, θ'_p and θ''_p are given by

$$\theta_p(r) = A_1 re^r + A_0 e^r$$

$$\theta'_p(r) = A_1 re^r + (A_1 + A_0)e^r, \text{ and}$$

$$\theta''_p(r) = A_1 re^r + (2A_1 + A_0)e^r$$

Substituting these expressions into the original differential equation yields

$$\begin{aligned}
&\theta''_p - 5\theta'_p + 6\theta_p \\
&= A_1 re^r + (2A_1 + A_0)e^r - 5\left[A_1 re^r + (A_1 + A_0)e^r\right] \\
&\quad + 6\left(A_1 re^r + A_0 e^r\right) \\
&= 2A_1 re^r + (2A_0 - 3A_1)e^r \\
&= re^r
\end{aligned}$$

By equating coefficients, we obtain

$$2A_1 = 1 \quad \Rightarrow \quad A_1 = \frac{1}{2}$$

$$2A_0 - 3A_1 = 0 \Rightarrow \quad A_0 = \frac{3}{4}$$

Therefore, a particular solution of the non-homogeneous differential equation

$$\theta''_p - 5\theta'_p + 6\theta_p = re^r$$

is given by

$$\theta_p(r) = \frac{re^r}{2} + \frac{3e^r}{4}$$

17. $y''(t) - 3y'(t) + 2y(t) = e^t \sin t$

Solution:

The corresponding homogeneous equation is

$$y'' - 3y' + 2y = 0$$

which has the associated auxiliary equation

$$r^2 - 3r + 2 = 0$$

This auxiliary equation has roots $r = 1,2$.

Thus, a general solution of this homogeneous equation is given by

$$y_h(t) = C_1 e^t + C_2 e^{2t}$$

The non-homogenous term of the original differential equation is $e^t \sin t$. According to Table 4.1 on page 208 of the text, this non-homogeneous term is of Type VI with $\alpha = 1$, $\beta = 1$, $a = 0$, and $b = 1$. Thus, a particular solution of the non-homogeneous differential equation will have the form

$$y_p(t) = t^s e^t (A\cos t + B\sin t).$$

Since neither $e^t \cos t$ nor $e^t \sin t$ is a solution of the corresponding homogeneous equation, we let $s = 0$.

Therefore, y_p, y'_p and y''_p are given by

$$\begin{aligned}
y_p(t) &= e^t (A\cos t + B\sin t) \\
y'_p(t) &= e^t (A\cos t + B\sin t) \\
&\quad + e^t(-A\sin t + B\cos t) \\
&= e^t\left[(A+B)\cos t + (-A+B)\sin t\right] \\
y''_p(t) &= e^t\left[(A+B)\cos t + (-A+B)\sin t\right] \\
&\quad + e^t\left[-(A+B)\sin t + (-A+B)\cos t\right] \\
&= e^t\left[2B\cos t - 2A\sin t\right]
\end{aligned}$$

Substituting these expressions into the original differential equation yields

$$\begin{aligned}
&y''_p - 3y'_p + 2y_p \\
&= e^t(2B\cos t - 2A\sin t) \\
&\quad - 3\left\{e^t\left[(A+B)\cos t + (-A+B)\sin t\right]\right\} \\
&\quad + 2\left[e^t(A\cos t + B\sin t)\right] \\
&= (-B-A)e^t \cos t + (A-B)e^t \sin t \\
&= e^t \sin t
\end{aligned}$$

By equating coefficients, we obtain

$$-A - B = 0 \quad \text{and} \quad A - B = 1$$

$$\Rightarrow \quad B = -\frac{1}{2} \quad \text{and} \quad A = \frac{1}{2}$$

Therefore, a particular solution to the non-homogeneous differential equation

$$y'' - 3y' + 2y = e^t \sin t$$

is given by

$$y_p(t) = e^t \frac{\cos t - \sin t}{2}$$

Thus, a general solution of the original, non-homogeneous differential equation is

$$y(t) = y_h(t) + y_p(t)$$
$$= C_1 e^t + C_2 e^{2t} + e^t \frac{\cos t - \sin t}{2}$$

31. $y''(\theta) - y(\theta) = \sin\theta - e^{2\theta}$ □
$$y(0) = 1 \;\square\; y'(0) = -1$$

Solution:

We first solve the associated homogeneous equation

$$y'' - y = 0$$

and obtain as general solution

$$y_h(\theta) = C_1 e^{\theta} + C_2 e^{-\theta}$$

Next we will use the superposition principle and consider separately the equations

$$y'' - y = \sin\theta \qquad (3)$$

and

$$y'' - y = -e^{2\theta} \qquad (4)$$

For equation (3), according to Table 4.1 on page 208 of the text, this non-homogeneous term is of Type III (with $a = 0$, $b = 1$, and $\beta = 1$). Thus, we want a particular solution of equation (3) to have the form

$$y_{p1}(\theta) = A\cos\theta + B\sin\theta .$$

Therefore, y_{p1} and y''_{p1} are given by

$$y_{p1}(\theta) = A\cos\theta + B\sin\theta$$
$$y''_{p1}(\theta) = -A\cos\theta - B\sin\theta$$

Substituting these expressions into equation (3) yields

$$y''_{p1} - y_{p1}$$
$$= -A\cos\theta - B\sin\theta - \left(A\cos\theta + B\sin\theta\right)$$
$$= -2A\cos\theta - 2B\sin\theta$$
$$= \sin\theta$$

By equating coefficients, we obtain

$$-2A = 0 \qquad \Rightarrow A = 0$$
$$-2B = 1 \qquad \Rightarrow B = -\frac{1}{2}$$

Therefore, a particular solution of the non-homogeneous differential equation (3) is

$$y_{p1}(\theta) = -\frac{\sin\theta}{2}$$

For equation (4), according to Table 4.1 on page 208 of the text, this non-homogeneous term is of Type II (with $a = -1$ and $\alpha = 2$). Thus, we want a particular solution of equation (4) to have the form

$$y_{p2}(\theta) = De^{2\theta} .$$

Therefore, y_p and y''_p are given by

$$y_{p2}(\theta) = De^{2\theta}$$
$$y''_{p2}(\theta) = 4De^{2\theta}$$

Substituting these expressions into equation (4) yields

$$y''_{p2} - y_{p2}$$
$$= 4De^{2\theta} - De^{2\theta}$$
$$= 3De^{2\theta}$$
$$= -e^{2\theta}$$

By equating coefficients, we obtain

$$3D = -1 \qquad \Rightarrow D = -\frac{1}{3}$$

Therefore, a particular solution of the non-homogeneous differential equation (4) is

$$y_{p2}(\theta) = -\frac{e^{2\theta}}{3}$$

It follows from the superposition principle that a general solution to the original equation is given by

$$y(\theta) = y_h(\theta) + y_{p1}(\theta) + y_{p2}(\theta)$$
$$= C_1 e^{\theta} + C_2 e^{-\theta} - \frac{\sin\theta}{2} - \frac{e^{2\theta}}{3}$$

Thus we have

$$y'(\theta) = C_1 e^{\theta} - C_2 e^{-\theta} - \frac{\cos\theta}{2} - \frac{2e^{2\theta}}{3}$$

The initial conditions give

$$y(0) = 1 \qquad \Rightarrow C_1 + C_2 - \frac{1}{3} = 1$$
$$y'(0) = -1 \Rightarrow C_1 - C_2 - \frac{1}{2} - \frac{2}{3} = -1$$

Thus we have $C_1 = \frac{3}{4}$ and $C_2 = \frac{7}{12}$

Therefore the solution is

$$y(\theta) = \frac{3e^{\theta}}{4} + \frac{7e^{-\theta}}{12} - \frac{\sin\theta}{2} - \frac{e^{2\theta}}{3}$$

5. Text – Section 4.9

2. $y'' + y = \sec x$

Solution:

Step 1) solve complimentary equation

$$y'' + y = 0$$

$$r = \frac{0 \pm \sqrt{0 - 4(1)(1)}}{2} = \pm i$$

$$y_c = C_1 \cos x + C_2 \sin x$$

$$y_1 = \cos x \qquad y_2 = \sin x$$

Step 2) General form of solution

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = \int \frac{-y_2 f(x)}{w(y_1, y_2)} dx$$

$$u_2 = \int \frac{y_1 f(x)}{w(y_1, y_2)} dx$$

$$w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x - (-\sin^2 x) = 1$$

$$u_1 = \int \frac{-\sin x \sec x}{1} dx = -\int \sin x \sec x dx$$

$$= -\int \tan x dx = \ln|\cos x|$$

$$u_2 = \int \frac{\cos x \sec x}{1} dx = \int \cos x \sec x dx$$

$$= \int dx = x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \cos x \ln|\cos x| + t \sin x$$

Thus the general solution is

$$y = \cos x \ln|\cos x| + x \sin x$$

$$+ C_1 \cos x + C_2 \sin x$$

6. $y'' + 2y' + y = e^{-x}$

Solution:

Step 1) solve complimentary equation

$$y'' + 2y' + y = 0$$

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(1)}}{2} = -1$$

$$y_c = C_1 e^{-x} + C_2 x e^{-x}$$

$$y_1 = e^{-x} \qquad y_2 = x e^{-x}$$

Step 2) General form of solution

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = \int \frac{-y_2 f(x)}{w(y_1, y_2)} dx$$

$$u_2 = \int \frac{y_1 f(x)}{w(y_1, y_2)} dx$$

$$w(y_1, y_2) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}$$

$$= e^{-x}\left(e^{-x} - xe^{-x}\right) - xe^{-x}(-e^{-x})$$

$$= e^{-2x}$$

$$u_1 = \int \frac{-xe^{-x}e^{-x}}{e^{-2x}} dx = -\int x dx = -\frac{x^2}{2}$$

$$u_2 = \int \frac{e^{-x}e^{-x}}{e^{-2x}} dx = \int dx = x$$

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{x^2}{2} e^{-x} + x^2 e^{-x} = \frac{x^2}{2} e^{-x}$$

Thus the general solution is

$$y = \frac{x^2}{2} e^{-x} + C_1 e^{-x} + C_2 x e^{-x}$$

17. $y'' + y = 3\sec x - x^2 + 1$

Solution:

Step 1) solve complimentary equation

$$y'' + y = 0$$

$$r = \frac{0 \pm \sqrt{0 - 4(1)(1)}}{2} = \pm i$$

$$y_c = C_1 \cos x + C_2 \sin x$$

$$y_1 = \cos x \qquad y_2 = \sin x$$

Step 2) General form of solution

We will use the superposition principle and consider separately the equations

$$y'' + y = 3\sec x \qquad (5)$$

and

$$y'' + y = -x^2 + 1 \qquad (6)$$

For equation (5)

$$y_{p1} = u_1 y_1 + u_2 y_2$$

$$u_1 = \int \frac{-y_2 f(x)}{w(y_1, y_2)} dx$$

$$u_2 = \int \frac{y_1 f(x)}{w(y_1, y_2)} dx$$

$$w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x - (-\sin^2 x) = 1$$

$$u_1 = \int \frac{-3\sin x \sec x}{1} dx = -3\int \tan x dx$$

$$= 3\ln|\cos x|$$

$$u_2 = \int \frac{3\cos x \sec x}{1} dx = 3\int dx = 3x$$

$$y_{p1} = u_1 y_1 + u_2 y_2$$

$$= 3\cos x \ln|\cos x| + 3x\sin x$$

We can use the method of undetermined coefficients to solve equation (6).

According to Table 4.1 on page 208 of the text, this non-homogeneous term is of Type I. Thus, we want a particular solution of this differential equation to have the form

$$y_{p2}(x) = A_2 x^2 + A_1 x + A_0 .$$

Therefore, y_{p2} and y''_{p2} are given by

$$y_{p2}(x) = A_2 x^2 + A_1 x + A_0$$

$$y''_{p2}(x) = 2A_2$$

Substituting these expressions into equation (6) yields

$$y'' + y$$

$$= 2A_2 + \left(A_2 x^2 + A_1 x + A_0\right)$$

$$= A_2 x^2 + A_1 x + 2A_2 + A_0$$

$$= -x^2 + 1$$

By equating coefficients, we obtain

$$A_2 = -1$$

$$A_1 = 0$$

$$2A_2 + A_0 = 1 \quad \Rightarrow A_0 = 3$$

Therefore, a particular solution of equation (6) is given by

$$y_{p2}(x) = -x^2 + 3$$

Thus the general solution is

$$y = y_c + y_{p1} + y_{p2}$$

$$= C_1 \cos x + C_2 \sin x + 3\cos x \ln|\cos x|$$

$$+ 3x\sin x - x^2 + 3$$

21. $x^2 z'' - xz' + z = x\left(1 + \dfrac{3}{\ln x}\right)$

Solution:

The given differential equation is a Cauchy-Euler equation. We make the substitution $x = e^t$ to transform the Cauchy-Euler equation into an equation with constant coefficients given by

$$z'' - 2z' + z = e^t\left(1 + \frac{3}{t}\right)$$

Step 1) solve complimentary equation

$$z'' - 2z' + z = 0$$

$$r = \frac{2 \pm \sqrt{2^2 - 4(1)(1)}}{2} = 1$$

$$z_c = C_1 e^t + C_2 t e^t$$

$$z_1 = e^t \qquad z_2 = te^t$$

Step 2) General form of solution

$$z_p = u_1 z_1 + u_2 z_2$$

$$u_1 = \int \frac{-z_2 f(t)}{w(z_1, z_2)} dt$$

$$u_2 = \int \frac{z_1 f(t)}{w(z_1, z_2)} dt$$

$$w(z_1, z_2) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$$= e^t\left(e^t + te^t\right) - te^t(e^t) = e^{2t}$$

$$u_1 = \int \frac{-te^t e^t\left(1 + \frac{3}{t}\right)}{e^{2t}} dt$$

$$= -\int (t+3)dt = -\frac{t^2}{2} - 3t$$

$$u_2 = \int \frac{e^t e^t\left(1 + \frac{3}{t}\right)}{e^{2t}} dt$$

$$= \int\left(1 + \frac{3}{t}\right)dt = t + 3\ln|t|$$

Thus,

$$z_p = u_1 z_1 + u_2 z_2$$

$$= \left(-\frac{t^2}{2} - 3t \right) e^t + \left(t + 3 \ln|t| \right) t e^t$$

$$= \left(\frac{t^2}{2} - 3t + 3t \ln|t| \right) e^t$$

Thus the general solution is

$$z_p = C_1 e^t + C_2 t e^t + \left(\frac{t^2}{2} - 3t + 3t \ln|t| \right) e^t$$

$$= C_1 e^t + (C_2 - 3) t e^t + \frac{1}{2} t^2 e^t + 3 t e^t \ln|t|$$

With $x = e^t$ (so that $t = \ln|x|$), $c_1 = C_1$ and $c_2 = C_2 - 3$, the general solution can be expressed as follows

$$z_p = c_1 x + c_2 x \ln|x| + \frac{1}{2} x (\ln|x|)^2$$
$$+ 3x(\ln|x|)[\ln(\ln|x|)]$$