Assignment #5

Section 4.2

#3) $xy'' - yy' = \sin x$

$$\Rightarrow L[y] = xy'' - yy'$$

Check linearity: sub in $L[cy_1]$:

$$\left.\begin{aligned} y &= cy_1 \\ y' &= c \\ y'' &= 0 \end{aligned}\right\} \quad \therefore L[cy_1] = 0 - c^2y' = -c^2y'$$

Sub in $cL[y_1]$

$$\left.\begin{aligned} y &= y_1 \\ y' &= 1 \\ y'' &= 0 \end{aligned}\right\} \quad \therefore cL[y_1] = 0 - cy' = -cy'$$

Since $L[cy_1] \neq cL[y_1]$, the function is non-linear

#6) $\frac{d^2\theta}{dx^2} = \cos\theta$

$$\Rightarrow L[\theta] = \theta'' - \cos\theta$$

Check Linearity: sub in $L[c\theta_1]$

$$\left.\begin{aligned} \theta &= c\theta_1 \\ \theta' &= c \\ \theta'' &= 0 \end{aligned}\right\} \quad \therefore L[c\theta_1] = -\cos(c\theta_1)$$

Sub in $cL[\theta_1]$

$$\left.\begin{aligned} \theta &= \theta_1 \\ \theta' &= 1 \\ \theta'' &= 0 \end{aligned}\right\} \quad \therefore cL[\theta_1] = -c\cos(\theta_1)$$

Since $L[c\theta_1] \neq cL[\theta_1]$, The function is non-linear

13a) $y'' - 4y' + 5y = 0 = L[y]$

$$\left.\begin{aligned} y_1 &= e^{2x}\cos x \\ y_2 &= e^{2x}\sin x \end{aligned}\right\} \text{ Given general solutions}$$

Find a solution which satisfies $y(0) = 2$ and $y'(0) = 1$

$\Rightarrow$ We know that if y_1 and y_2 are solutions, then $Ay_1 + By_2$ is also a solution if $L[y]$ is linear. $\therefore$ check to see if $L[y]$ is linear:

check 1

sub in $L[cy_1]$: $\left.\begin{aligned} y &= cy_1 \\ y' &= c \\ y'' &= 0 \end{aligned}\right\} L[cy_1] = -4c + 5cy_1$

sub in $cL[y_1]$: $\left.\begin{aligned} y &= y_1 \\ y' &= 1 \\ y'' &= 0 \end{aligned}\right\} cL[y_1] = -4c + 5cy_1$

$\therefore$ Since $L[cy_1] = cL[y_1]$ the function can be linear

check 2

sub in $L[y_1]$: $\left.\begin{aligned} y &= y_1 \\ y' &= 1 \\ y'' &= 0 \end{aligned}\right\} L[y_1] = -4 + 5y_2$

sub in $L[y_2]$: $\left.\begin{aligned} y &= y_2 \\ y' &= 1 \\ y'' &= 0 \end{aligned}\right\} L[y_2] = -4 + 5y_2$

sub in $L[y_1 + y_2]$: $\left.\begin{aligned} y &= y_1 + y_2 \\ y' &= 2 \\ y'' &= 0 \end{aligned}\right\} \begin{aligned} L[y_1 + y_2] &= -4(2) + 5y_1 + 5y_2 \\ &= -8 + 5y_1 + 5y_2 \end{aligned}$

$\therefore$ Since $L[y_1] + L[y_2] = L[y_1 + y_2]$, and $L[cy_1] = cL[y_1]$, The function is linear.

Now we can find A and B (constants) such that $Ay_1 + By_2$ is a solution that satisfies the boundary conditions.

$$\begin{aligned} y_3 &= Ay_1 + By_2 = Ae^{2x}\cos x + Be^{2x}\sin x \\ y_3' &= 2Ae^{2x}\cos x - Ae^{2x}\sin x + 2Be^{2x}\sin x + Be^{2x}\cos x \\ &= e^{2x}\cos x\,(2A + B) + e^{2x}\sin x\,(2B - A) \end{aligned}$$

$\therefore\ 2 = Ae^{2(0)}\cos(0) + Be^{2(0)}\cancelto{0}{\sin(0)} = A \qquad \boxed{\therefore A = 2}$

$\therefore\ 1 = e^{2(0)}\cos(0)(2A + B) + e^{2(0)}\cancelto{0}{\sin(0)}(2B - A) = 2A + B \qquad \boxed{\therefore B = -3}$

$\boxed{\therefore \text{The solution is } 2e^{2x}\cos x - 3e^{2x}\sin x}$

Section 4.3

#9) $y'' - 2y' + 5y = 0$

solutions: $y_1 = e^x \cos 2x$ initial conditions: $y(0) = 2$

$y_2 = e^x \sin 2x$ $y'(0) = 0$

a) Verify linear Independence: (check that Wronskian $\neq 0$)

$$\text{Wronskian} = W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$y_1 = e^x \cos 2x$ $y_2 = e^x \sin 2x$

$y_1' = e^x \cos 2x - 2e^x \sin 2x$ $y_2 = e^x \sin 2x + 2e^x \cos 2x$

$$W = e^x e^x \left[\cancel{\cos 2x \sin 2x} + 2\cos 2x \cos 2x\right] - e^x e^x\left[\cancel{\sin 2x \cos 2x} - 2\sin 2x \sin 2x\right]$$
$$= e^{2x}\left[2(\cos 2x)^2 + 2(\sin 2x)^2\right]$$
$$= 2e^{2x}\left[(\cos 2x)^2 + (\sin 2x)^2\right]$$
$$= 2e^{2x}[1]$$
$$= 2e^{2x}$$

$\Leftarrow$ can never $= 0$, so the two solutions must be linearly independe

b) Find a General Solution:

$y_3 = Ay_1 + By_2 = Ae^x \cos 2x + Be^x \sin 2x$ $\Leftarrow$ linear combination of both solutions.

c) Find the specific solution

$y_3 = Ae^x \cos 2x + Be^x \sin 2x$

$y_3' = Ae^x \cos 2x - 2Ae^x \sin 2x + Be^x \sin 2x + 2Be^x \cos 2x = e^x \cos(2x)\left[A + 2B\right] + e^x \sin(2x)\left[B - 2A\right]$

$y(0) = 2$ $2 = Ae^0 \cos(0) + Be^0 \sin(0) = A$ $\boxed{\therefore A = 2}$

$y'(0) = 0$ $0 = e^0 \cos(0)[A + 2B] + e^0 \sin(0)[B - 2A] = A + 2B$ $\boxed{\therefore B = -1}$

$\boxed{\therefore \text{The specific solution is } y = e^x(2\cos 2x - \sin 2x)}$

#12) $y'' - y = 0$

solutions: $y_1 = \cosh x$ initial conditions: $y(0) = 1$

$y_2 = \sinh x$ $y'(0) = -1$

a) Verify linear independence

$y_1 = \cosh x$ $y_2 = \sinh x$

$y_1' = \sinh x$ $y_2' = \cosh x$

$$
\begin{aligned}
W &= y_1 y_2' - y_2 y_1' \\
&= (\cosh x)^2 - (\sinh x)^2 \\
&= 1 - (\sinh x)^2 - (\sinh x)^2 \\
&= 1 - 2(\sinh x)^2 \Leftarrow \text{not always 0, so linearly independent.}
\end{aligned}
$$

b) Find a general solution

$A y_1 + B y_2 = 0$

$A \cosh x + B \sinh x = 0$ $\Leftarrow$ general solution

c) Find specific solution

$y_3 = A \cosh(x) + B \sinh(x)$

$y_3' = A \sinh(x) + B \cosh(x)$

$y(0) = 1$ $1 = A \cosh(0) + B \sinh(0) = A$ $\boxed{\therefore A = 1}$

$y'(0) = -1$ $-1 = A \sinh(0) + B \cosh(0) = B$ $\boxed{\therefore B = -1}$

$\therefore$ The specific solution is $\boxed{y_3 = \cosh(x) - \sinh(x)}$

21) $y'' + py' + qy = 0$

Compute the Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_2 y_1'$$

if y_1 and y_2 are linearly independent, then $W \neq 0$

$$\therefore \ c = y_1 y_2' - y_2 y_1' \qquad (c \neq 0)$$

if $y_1(x_0) = 0$, then $c = -y_2 y_1'$ Therefore y_2 and $y_1' \neq 0$.
if $y_2(x_0) = 0$, then $c = y_1 y_2'$ Therefore y_1 and $y_2' \neq 0$

$\boxed{\therefore \ y_1 \text{ and } y_2 \text{ cannot BOTH be } \emptyset \text{ at any } x \text{ in } (a,b)}$

Section 4.5

#7) $2u'' + 7u' - 4u = 0$

$2m^2 + 7m - 4 = 0$ (where $u = e^{rx}$, $u' = re^{rx}$, $u'' = r^2 e^{rx}$)

$$m = \frac{-7 \pm \sqrt{7^2 - 4(2)(-4)}}{2(2)}$$

$$m = \frac{-7 \pm \sqrt{49 + 32}}{4}$$

$$m = \frac{-7 \pm 9}{4} = \frac{1}{2} \text{ or } -4$$

$\boxed{\therefore \text{ The general solution is } \quad y = Ae^{\frac{1}{2}x} + Be^{-4x}}$

#17) $z'' - 2z' - 2z = 0 \qquad z(0)=0 \qquad z'(0)=3$

$$m^2 - 2m - 2 = 0$$

$$m = \frac{2 \pm \sqrt{2^2 - 4(-2)}}{2}$$

$$m = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$$

$\therefore m = 1+\sqrt{3}$ or $m = 1-\sqrt{3}$

$\therefore$ The solution is $z = Ae^{(1+\sqrt{3})x} + Be^{(1-\sqrt{3})x}$

$$z' = (1+\sqrt{3})Ae^{(1+\sqrt{3})x} + (1-\sqrt{3})Be^{(1-\sqrt{3})x}$$

$z(0)=0 \qquad 0 = Ae^{(1+\sqrt{3})(0)} + Be^{(1-\sqrt{3})(0)} = A + B \quad ①$

$z'(0)=3 \qquad 3 = (1+\sqrt{3})Ae^{(1+\sqrt{3})(0)} + (1-\sqrt{3})Be^{(1-\sqrt{3})(0)} = (1+\sqrt{3})A + (1-\sqrt{3})B \quad ②$

$① \rightarrow ② \qquad 3 = (1+\sqrt{3})(-B) + (1-\sqrt{3})(B)$

$$3 = \cancel{-B} - \sqrt{3}B \cancel{+B} - \sqrt{3}B$$

$$3 = -2\sqrt{3}\,B$$

$$B = \frac{-3}{2\sqrt{3}} = \frac{-3\sqrt{3}}{2(3)} = -\frac{\sqrt{3}}{2}$$

$$\therefore A = \frac{\sqrt{3}}{2}$$

$\boxed{\therefore \text{The specific solution is } z = \frac{\sqrt{3}}{2}e^{(1+\sqrt{3})x} - \frac{\sqrt{3}}{2}e^{(1-\sqrt{3})x}}$

18) $y'' - 6y' + 9y = 0 \qquad y(0) = 2 \qquad y'(0) = 25/3$

$m^2 - 6m + 9 = 0$
$(m-3)^2 = 0$
$\therefore m = 3$

Since there are 2 equal roots, we use the form $y = Ae^{mx} + Bxe^{mx}$
$\therefore y = Ae^{3x} + Bxe^{3x}, \quad y' = 3Ae^{3x} + Be^{3x} + 3Bxe^{3x}$

$y(0) = 2 \quad 2 = Ae^0 + B(0)e^0 = A \quad \boxed{\therefore A = 2}$
$y'(0) = 25/3 \quad \frac{25}{3} = 3Ae^0 + Be^0 + 3B(0)e^0 = 3A + B \quad \boxed{\therefore B = \frac{7}{3}}$

$\boxed{\therefore \text{The specific solution is } y = 2e^{3x} + \frac{7}{3}xe^{3x}}$

Section 4.6

#7) $4y'' - 4y' + 26y = 0$

$4m^2 - 4m + 26 = 0$

$m = \dfrac{4 \pm \sqrt{16 - 4(4)(26)}}{4(2)}$

$m = \dfrac{4 \pm \sqrt{-400}}{8} = \dfrac{4 \pm 20i}{8} = \frac{1}{2} \pm \frac{5}{2}i$

$\boxed{\therefore y = Ae^{x/2}\cos\left(\frac{5}{2}x\right) + Be^{x/2}\sin\left(\frac{5}{2}x\right)}$

#24) $y'' + 9y = 0 \qquad y(0)=1 \qquad y'(0)=1$

$m^2 + 9 = 0$

$\therefore m^2 = -9$

$\therefore m = 3i \qquad \therefore y = Ae^{0}\sin(3x) + Be^{0}\cos(3x)$

$y = A\sin(3x) + B\cos(3x)$

$y(0)=1 \qquad 1 = A\sin(0) + B\cos(0) = B \qquad \boxed{\therefore B = 1}$

$y'(0)=1 \qquad 1 = 3A\cos(0) - 3B\sin(0) = 3A \qquad \boxed{\therefore A = 1/3}$

$$\boxed{\therefore y = \tfrac{1}{3}\sin(3x) + \cos(3x)}$$

#25) $y'' - 2y' + 2y = 0 \qquad y(\pi) = e^{\pi} \qquad y'(\pi) = 0$

$m^2 - 2m + 2 = 0$

$m = 2 \pm \sqrt{4 - 4(2)}\,/2$

$m = 2 \pm \sqrt{-4}\,/2 \qquad \therefore y = Ae^{x}\sin x + Be^{x}\cos x$

$m = 2 \pm 2i/2 = 1 \pm i \qquad \therefore y' = Ae^{x}\sin x + Ae^{x}\cos x + Be^{x}\cos x - Be^{x}\sin x$

$y(\pi) = e^{\pi} \qquad e^{\pi} = Ae^{\pi}\overset{0}{\cancel{\sin\pi}} + Be^{\pi}\overset{-1}{\cancel{\cos\pi}} = -Be^{\pi} \qquad \boxed{\therefore B = -1}$

$y'(\pi) = 0 \qquad 0 = e^{\pi}\overset{0}{\cancel{\sin\pi}}(A - B) + e^{\pi}\overset{-1}{\cancel{\cos\pi}}(A + B) = -e^{\pi}(A + B) \qquad \boxed{\therefore A = 1}$

$$\boxed{\therefore y = e^{x}\sin x - e^{x}\cos x}$$