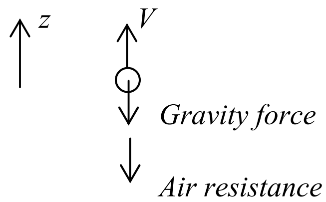
Problem 1.



Solution:

1. Identify variables: dependent variable = instantaneous velocity, V [*m/s*]
   independent variable = time, t [*s*]

2. Fundamental law:
   Newton's Second Law:

$$\sum F_z = m\frac{dV}{dt}$$

   a) Gravity force $F_g = -m \cdot g$

   b) Air resistance $F_a = -k \cdot V$

   First order, linear separable ODE

$$m\frac{dV}{dt} = -kV - mg$$

$$\frac{dV}{dt} + \frac{k}{m}V = -g$$

   Initial condition

$$V(t=0) = V_0 [m/s]$$

3. Solve ODE
   The solution to the ODE is,

$$V = Ce^{-\frac{k}{m}t} - \frac{m}{k}g$$

   According to the initial solution, the constant is,

$$C = V_0 + \frac{m}{k}g$$

   The final solution is

$$V = \left(V_0 + \frac{m}{k}g\right)e^{-\frac{k}{m}t} - \frac{m}{k}g$$

4. Maximum height
   When $V = 0$, the ball will reach the maximum height. The length of time is,

$$t_{\max} = \frac{m}{k}\ln\left(1 + \frac{kV_0}{mg}\right)$$

   The height of the ball can be determined by integrating velocity ($V = dz/dt$) with respect to t,

$$z(t) = \int_0^t V(t)dt$$

   The maximum height is

$$z_{\max} = \int_0^{t_{\max}} V(t)dt = \int_0^{t_{\max}} \left[\left(V_0 + \frac{m}{k}g\right)e^{-\frac{k}{m}t} - \frac{m}{k}g\right]dt$$

$$= \left[-\frac{m}{k}\left(V_0 + \frac{m}{k}g\right)e^{-\frac{k}{m}t} - \frac{m}{k}g\cdot t\right]_0^{t=\frac{m}{k}\ln\left(1+\frac{kV_0}{mg}\right)}$$

$$= \frac{m}{k}V_0 - \frac{m^2}{k^2}g\ln\left(1+\frac{kV_0}{mg}\right)$$

Problem 2.

Solution:

1. Identify variables: dependent variable = depth of water, y [m]
   independent variable = time, t [s]

2. Fundamental laws:
   Conservation of Mass:
   Change in Tank Volume = - Outflow rate of water

$$\frac{dV}{dt} = -Q_{ex}$$

   Tank volume

$$dV = A(y)dy = \pi\cdot r^2(y)dy = \pi\frac{R_0^2}{y_0}ydy$$

   Torricelli's Law

$$V_{ex} = C_d\sqrt{2\cdot g\cdot y}$$

   where C_d is the discharge coefficient (constant)

   Volume outflow rate of water

$$Q_{ex} = A_{ex}V_{ex} = \pi r_h^2 C_d\sqrt{2\cdot g\cdot y}$$

   First order, linear separable ODE

$$\pi\frac{R_0^2}{y_0}y\frac{dy}{dt} = -\pi r_h^2 C_d\sqrt{2\cdot g\cdot y}$$

$$\sqrt{y}dy = -C_d\sqrt{2g}\,y_0\frac{r_h^2}{R_0^2}dt$$

   Initial condition

$$y(t=0) = y_0[m]$$

3. Solve ODE
   The solution to the ODE is,

$$y^{3/2} = -\frac{3\sqrt{2g}}{2}C_d\,y_0\frac{r_h^2}{R_0^2}t + C$$

According to the initial solution, the constant is,

$$C = y_0^{3/2}$$

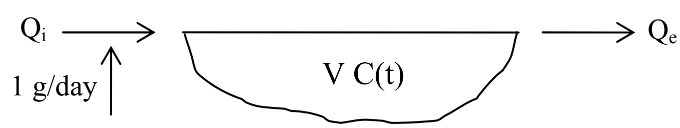
The final solution is

$$y^{3/2} = -\frac{3\sqrt{2g}}{2} C_d y_0 \frac{r_h^2}{R_0^2} t + y_0^{3/2}$$

The length of time it takes the tank to drain ($y = 0$) is

$$t = \frac{\sqrt{2} R_0^2 \sqrt{y_0}}{3\sqrt{g} \cdot C_d r_h^2}$$

Problem 3.



Solution:

1. Identify variables: dependent variable = concentration of X, C [g/*m*³]
   independent variable = time, t [*day*]

2. Fundamental law:
   Conservation of Mass:
   Change of X in the Lake = Inflow of X – Outflow of X

$$\frac{d(V \cdot C)}{dt} = X_i - X_e$$

$$X_i = 1g/day$$

$$X_e = C \cdot Q_e$$

First order, linear separable ODE

$$V\frac{dC}{dt} = 1 - CQ_e$$

where $Q_e = 50000[m^3/day]$, $V = 10^7[m^3]$

Initial condition

$$C(t = 0) = 10^{-3}[g/m^3]$$

3. Solve ODE
   The solution to the ODE is,

$$C = \frac{1}{Q_e} + Ae^{-\frac{Q_e}{V}t}$$

Substituting $Q_e = 50000[m^3/day]$, $V = 10^7[m^3]$ into the solution gives

$$C = 2\times10^{-5} + Ae^{-5\times10^{-3}t}$$

According to the initial solution, the constant is,

$$A = 9.8 \times 10^{-4}$$

The final solution is

$$C = 2 \times 10^{-5} + 9.8 \times 10^{-4} e^{-5\times10^{-3} t}$$

The length of time it takes for the lake concentration to drop to a level of $10^{-4}[g/m^3]$

$$t \approx 502[day]$$

Problem 4.

Solution:

1. Identify variables: dependent variable = current flow, I [A]
   independent variable = time, t [s]

2. Fundamental law:
   Kirchoff's Law:

$$\mathrm{E} = \sum \text{voltage drops}$$

$$E = L\frac{dI}{dt} + IR$$

First order, linear separable ODE

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

Initial condition

$$I(t=0) = 0$$

3. Solve ODE
   The solution to the ODE is,

$$I = \frac{E}{R} - Ce^{-\frac{R}{L}t}$$

According to the initial solution, the constant is,

$$C = \frac{E}{R}$$

The final solution is

$$I = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

When $t \to \infty$, the steady state value of the current is $I = \dfrac{E}{R}$

The length of time it takes for the current to achieve half its steady-state value is

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$t = \frac{L}{R}\ln 2 = \frac{0.1}{100}\ln 2 \approx 6.9 \times 10^{-4}[s]$$

Problem 5.

Solution:
Recall: the dimensional ODE from problem 4 is

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E}{L}$$

Let $I^* = I / I_{ref}$ and $t^* = t / t_{ref}$, thus,

$$\frac{dI}{dt} = \frac{I_{ref}}{t_{ref}} \frac{dI^*}{dt^*}$$

$$\frac{R}{L} I = \frac{R}{L} I_{ref} \cdot I^*$$

The non-dimensional differential equation is

$$\frac{dI^*}{dt^*} + \frac{R}{L} t_{ref} \cdot I^* = \frac{t_{ref}}{I_{ref}} \frac{E}{L}$$

Let $\frac{R}{L} t_{ref} = 1$, we have $t_{ref} = \frac{L}{R}$

Let $\frac{t_{ref}}{I_{ref}} \frac{E}{L} = 1$, we have $I_{ref} = \frac{E}{R}$

Thus, we have

$$\frac{dI^*}{dt^*} + I^* = 1$$

Initial condition

$$I^*\left(t^* = 0\right) = 0$$

The solution to the non-dimensional ODE is

$$I^* = 1 - Ce^{-t^*}$$

The constant can be determined according to the initial condition,

$$C = 1$$

The final solution is

$$I^* = 1 - e^{-t^*}$$

The solution in dimensional form

$$\frac{I}{I_{ref}} = 1 - e^{-\frac{t}{t_{ref}}}$$

$$I = I_{ref}\left(1 - e^{-\frac{t}{t_{ref}}}\right) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$