ME203 PROBLEM SET #2

1. Text – Section 2.2

7. $\dfrac{dy}{dx}=\dfrac{1-x^2}{y^2}$

Solution:

Solve $\dfrac{dy}{dx}=\dfrac{1-x^2}{y^2}$ (1)

Rewrite the equation:

$$y^2dy=\left(1-x^2\right)dx$$

Integrating, we have

$$\int y^2dy=\int\left(1-x^2\right)dx$$

$$\Rightarrow \quad \frac{y^3}{3}=x-\frac{x^3}{3}+C_1$$

Solve the last equation for y gives

$$y=\left(3x-x^3+C\right)^{1/3}$$

12. $x\dfrac{dv}{dx}=\dfrac{1-4v^2}{3v}$

Solution:

Solve $x\dfrac{dv}{dx}=\dfrac{1-4v^2}{3v}$ (2)

Rewrite the equation:

$$\frac{3v}{1-4v^2}dv=\frac{1}{x}dx$$

Integrating, we have

$$\int\frac{3v}{1-4v^2}dv=-\frac{3}{8}\int\frac{d\left(1-4v^2\right)}{1-4v^2}=\int\frac{1}{x}dx$$

$$\Rightarrow \quad -\frac{3}{8}\ln\left|1-4v^2\right|=\ln|x|+C_1$$

The solution to equation (2) is given implicitly by

$$1-4v^2=Cx^{-8/3}$$

Solve the last equation for v gives

$$v=\pm\frac{1}{2}\sqrt{1-Cx^{-8/3}}$$

21. $\dfrac{dy}{dx}=2\sqrt{y+1}\cos x$, $y(\pi)=0$

Solution:

Solve $\dfrac{dy}{dx}=2\sqrt{y+1}\cos x$ (3)

Rewrite the equation:

$$\frac{dy}{2\sqrt{y+1}}=\cos x dx$$

Integrating, we have

$$\int\frac{1}{2\sqrt{y+1}}dy=\int\cos x dx$$

$$\Rightarrow \quad \sqrt{y+1}=\sin x+C$$

Substituting $x=\pi$ and $y(\pi)=0$ gives

$$1=\sin\pi+C \Rightarrow C=1$$

Thus,

$$\sqrt{y+1}=\sin x+1$$

and so

$$y=\left(\sin x+1\right)^2-1=\sin^2 x+2\sin x$$

29. $\dfrac{dy}{dx}=y^{1/3}$, $y(0)=0$

Solution:

Solve $\dfrac{dy}{dx}=y^{1/3}$ (4)

(a) Rewriting the equation gives,

$$y^{-1/3}dy=dx$$

Integrating, we have

$$\int y^{-1/3}dy=\int dx$$

$$\Rightarrow \quad \frac{3y^{2/3}}{2}=x+C_1$$

Solve the last equation for y gives

$$y=\left(\frac{2x}{3}+C\right)^{3/2}$$

This shows that $y=\left(\dfrac{2x}{3}+C\right)^{3/2}$ is a solution to equation (4).

(b) Substituting $x=0$ and $y(0)=0$ into the solution gives

$$0=\left(0+C\right)^{3/2} \Rightarrow C=0$$

Thus the solution for this initial value problem is $y=\left(2x/3\right)^{3/2}$ for $x\geq 0$

(c) Substituting constant function $y\equiv 0$ into equation (4):

$$\frac{dy}{dx} = y^{1/3}$$

The left hand side is: $\frac{dy}{dx} = 0$

The right hand side is: $y^{1/3} = 0$

Thus, LHS = RHS

Also $y = 0$ when $x = 0$.

This shows that the constant function $y \equiv 0$ is also a solution to the initial value problem. Hence this initial value problem does not have a unique solution.

(d) $\frac{dy}{dx} = f(x, y) = y^{1/3}$

The conditions for a unique solution in theorem 1 are that f and $\partial f / \partial y$ are continuous function in a rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0).

In this problem, function f is continuous in a rectangle

$$R = \{(x, y) : -\infty < x < \infty, -\infty < y < \infty\}.$$

However

$$\frac{\partial f}{\partial y} = \frac{1}{3y^{2/3}}, \quad y \neq 0$$

We can find that $\frac{\partial f}{\partial y}$ is not continuous at $(0,0)$.

The conditions of Theorem 1 are not satisfied for this initial value problem.

34. $\frac{dT}{dt} = k(M - T)$

Solution:

Solve $\frac{dT}{dt} = k(M - T)$ (5)

(a) Rewriting the equation gives,

$$\frac{dT}{M - T} = kdt$$

Integrating, we have

$$\int \frac{dT}{M - T} = \int kdt$$

$$\Rightarrow \quad -\ln|M - T| = kt + C_1$$

Solve the last equation for T gives

$$T = M + Ce^{-kt}$$

(b) In this problem, since the initial value is $T(0) = 100$ and $M = 70$, we can solve the constant.

$$100 = 70 + Ce^0 \Rightarrow C = 30$$

Then the solution for this initial value problem is

$$T = 70 + 30e^{-kt}$$

Since after 6 min, the thermometer read 80°, we have,

$$80 = 70 + 30e^{-6k}$$

Solve the last equation gives

$$k = (\ln 3)/6 \approx 0.1831$$

Then after 20 min ($t = 20$), we have,

$$T = 70 + 30e^{-(\ln 3) \cdot 20/6} = 70.77^\circ$$

2. Text – Section 2.3

7. $\frac{dy}{dx} - y = e^{3x}$

Solution:

Solve $\frac{dy}{dx} - y = e^{3x}$ (6)

Here $P(x) = -1$, so

$$\int P(x)dx = \int (-1)dx = -x$$

Thus an integrating factor is,

$$\mu(x) = e^{-x}$$

Multiplying equation (6) by $\mu(x)$ yields

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{2x}$$

That is

$$\frac{d}{dx}\left(e^{-x}y\right) = e^{2x}$$

Integrate both sides and solve for y to find

$$e^{-x}y = \int e^{2x}dx = \frac{1}{2}e^{2x} + C$$

Thus

$$y = \frac{1}{2}e^{3x} + Ce^{x}$$

13. $y\frac{dx}{dy} + 2x = 5y^3$

Solution:

Solve $y\frac{dx}{dy}+2x=5y^3$ (7)

Standard Form:

$$\frac{dx}{dy}+\frac{2}{y}x=5y^2 \quad (8)$$

Here $P(y)=\frac{2}{y}$, so

$$\int P(y)dy=\int \frac{2}{y}dy=2\ln|y|$$

Thus an integrating factor is,

$$\mu(y)=e^{2\ln|y|}=y^2$$

Multiplying equation (8) by $\mu(y)$ yields

$$y^2\frac{dx}{dy}+2xy=5y^4$$

That is

$$\frac{d}{dy}\left(xy^2\right)=5y^4$$

Integrate both sides and solve for x to find

$$xy^2=\int 5y^4dy=y^5+C$$

Thus

$$x=y^3+Cy^{-2}$$

18. $\frac{dy}{dx}+4y-e^{-x}=0$, $y(0)=\frac{4}{3}$

Solution:

Solve $\frac{dy}{dx}+4y-e^{-x}=0$ (9)

Standard Form:

$$\frac{dy}{dx}+4y=e^{-x}$$

Here $P(x)=4$, so

$$\int P(x)dx=\int 4dx=4x$$

Thus an integrating factor is,

$$\mu(x)=e^{4x}$$

Multiplying equation (1) by $\mu(x)$ yields

$$e^{4x}\frac{dy}{dx}+4e^{4x}y=e^{3x}$$

That is

$$\frac{d}{dx}\left(e^{4x}y\right)=e^{3x}$$

Integrate both sides and solve for y to find

$$e^{4x}y=\int e^{3x}dx=\frac{1}{3}e^{3x}+C$$

Thus

$$y=\frac{1}{3}e^{-x}+Ce^{-4x}$$

Substituting $x=0$ and $y(0)=\frac{4}{3}$ gives

$$\frac{4}{3}=\frac{1}{3}e^{0}+Ce^{0}\Rightarrow C=1$$

Thus the solution is

$$y=\frac{1}{3}e^{-x}+e^{-4x}$$

3. Text – Section 2.4

9. $(2xy+3)dx+\left(x^2-1\right)dy=0$

Solution:

Solve $(2xy+3)dx+\left(x^2-1\right)dy=0$ (10)

Here $M(x,y)=2xy+3$ and $N(x,y)=x^2-1$.

Because

$$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x}$$

equation (10) is exact.

To find $F(x,y)$, we begin by integrating M with respect to x:

$$F(x,y)=\int(2xy+3)dx=x^2y+3x+g(y) \quad (11)$$

Differentiating $F(x,y)$ with respect to y gives

$$\frac{\partial F}{\partial y}(x,y)=N(x,y)$$

$$\Rightarrow \quad x^2+g'(y)=x^2-1$$

Thus $g'(y)=-1$, we can take $g(y)=-y$.

Hence, from (11), we have

$$F(x,y)=x^2y+3x-y$$

The solution to equation (10) is given implicitly by

$$x^2y+3x-y=C$$

Thus we have $y=(C-3x)/\left(x^2-1\right)$

15. $\cos\theta dr-\left(r\sin\theta-e^{\theta}\right)d\theta=0$

Solution:

Solve $\cos\theta dr - \left(r\sin\theta - e^{\theta}\right)d\theta = 0$ (12)

This differential equation is expressed in the variable r and θ. Since the variables x and y are dummy variables, this equation is solved in exactly the same way as an equation in x and y. We will look for a solution with independent variable θ and dependent variable r. We see that the differential equation is expressed in the differential form

$$M(r,\theta)dr + N(r,\theta)d\theta = 0$$

Here $M(r,\theta) = \cos\theta$ and $N(r,\theta) = -\left(r\sin\theta - e^{\theta}\right)$.

Because

$$\frac{\partial M}{\partial \theta} = -\sin\theta = \frac{\partial N}{\partial r}$$

equation (12) is exact.

To find $F(r,\theta)$, we begin by integrating M with respect to r:

$$F(r,\theta) = \int \cos\theta dr = r\cos\theta + g(\theta) \quad (13)$$

Next we take the partial derivative of (13)

$$\frac{\partial F}{\partial \theta}(r,\theta) = N(r,\theta)$$

$$\Rightarrow \quad -r\sin\theta + g'(\theta) = -\left(r\sin\theta - e^{\theta}\right)$$

Thus $g'(\theta) = e^{\theta}$, we can take $g(\theta) = e^{\theta}$.

Hence, from (13), we have

$$F(r,\theta) = r\cos\theta + e^{\theta}$$

The solution to equation (12) is given implicitly by

$$r\cos\theta + e^{\theta} = C$$

Thus we have

$$r = \left(C - e^{\theta}\right)/\cos\theta = \left(C - e^{\theta}\right)\sec\theta$$

24. $\left(e^t x + 1\right)dt + \left(e^t - 1\right)dx = 0$, $x(1) = 1$

Solution:

Solve $\left(e^t x + 1\right)dt + \left(e^t - 1\right)dx = 0$ (14)

Here $M(t,x) = e^t x + 1$ and $N(t,x) = e^t - 1$.

Because

$$\frac{\partial M}{\partial x} = e^t = \frac{\partial N}{\partial t}$$

equation (14) is exact.

To find $F(t,x)$, we begin by integrating M with respect to t:

$$F(t,x) = \int \left(e^t x + 1\right)dt = e^t x + t + g(x) \quad (15)$$

Next we take the partial derivative of (15)

$$\frac{\partial F}{\partial x}(t,x) = N(t,x)$$

$$\Rightarrow \quad e^t + g'(x) = e^t - 1$$

Thus $g'(x) = -1$, we can take $g(x) = -x$.

Hence, from (15), we have

$$F(t,x) = e^t x + t - x$$

The solution to equation (14) is given implicitly by

$$e^t x + t - x = C$$

Thus

$$x = (C - t)/\left(e^t - 1\right)$$

We can determine the constant according the initial condition, that is,

$$1 = (C-1)/(e-1) \Rightarrow C = e$$

Thus we have $x = (e - t)/\left(e^t - 1\right)$

4. Find the general solution to differential equation: $xdy - \left(y + x^3\right)dx = 0$**.**

Solution:

Solve $xdy - \left(y + x^3\right)dx = 0$ (16)

Expand dx term: $xdy - ydx - x^3 dx = 0$

Isolate $xdy - ydx$ terms: $(xdy - ydx) = x^3 dx$

Divide by x^2 to get form (1) from handout

$$\frac{xdy - ydx}{x^2} = xdx \quad (17)$$

From (1), substitute $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

$$d\left(\frac{y}{x}\right) = xdx$$

Solve by integrating both sides

$$\frac{y}{x} = \frac{x^2}{2} + C$$

$$\Rightarrow \quad y = \frac{x^3}{2} + Cx$$