CHAPTER 8: Series Solutions of Differential Equations

EXERCISES 8.1: Introduction: The Taylor Polynomial Approximation, page 452

3. Using the initial condition, $y(0)=0$, we substitute $x=0$ and $y=0$ into the given equation and find $y'(0)$.

$$y'(0)=\sin(0)+e^{0}=1.$$

To determine $y''(0)$, we differentiate the given equation with respect to x and substitute $x=0$, $y=0$, and $y'=1$ in the formula obtained:

$$y''=\left(\sin y+e^{x}\right)'=\left(\sin y\right)'+\left(e^{x}\right)'=y'\cos y+e^{x},$$

$$y''(0)=1\cdot\cos(0)+e^{0}=2.$$

Similarly, differentiating $y''(x)$ and substituting, we obtain

$$y'''=\left(y'\cos y+e^{x}\right)'=\left(y'\cos y\right)'+\left(e^{x}\right)'=y''\cos y+\left(y'\right)^2(-\sin y)+e^{x},$$

$$y'''(0)=y''(0)\cos y(0)+\left(y'(0)\right)^2[-\sin y(0)]+e^{0}=2\cdot\cos 0+(1)^2(-\sin 0)+1=3.$$

Thus the first three nonzero terms in the Taylor polynomial approximations to the solution of the given initial value problem are

$$y(x)=y(0)+\frac{y'(0)}{1!}x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\cdots$$

$$=0+\frac{1}{1}x+\frac{2}{2}x^2+\frac{3}{6}x^3+\cdots=x+x^2+\frac{1}{2}x^3+\cdots.$$

7. We use the initial conditions to find $y''(0)$. Writing the given equation in the form

$$y''(\theta)=-y(\theta)^3+\sin\theta$$

and substituting $\theta=0$, $y(0)=0$, we get

$$y''(0)=-y(0)^3+\sin 0=0.$$

Differentiating the given equation we obtain

$$y'''=\left(y''\right)'=-\left(y^3\right)'+(\sin\theta)'=-3y^2y'+\cos\theta$$

$$\Rightarrow\qquad y'''(0)=-3y(0)^2y'(0)+\cos 0=-3(0)^2(0)+1=1.$$

Similarly, we get

$$y^{(4)} = (y''')' = \left(-(y^3)' + \cos\theta\right)' = -3y^2 y'' - 6y(y')^2 - \sin\theta$$

$$\Rightarrow \quad y^{(4)}(0) = -3y(0)^2 y''(0) - 6y(0)(y'(0))^2 - \sin 0 = 0 .$$

To simplify further computations we observe that since the Taylor expansion for $y(\theta)$ has the form

$$y(\theta) = \frac{1}{3!}\theta^3 + \cdots,$$

then the Taylor expansion for $y(\theta)^3$ must begin with the term $(1/3!)^3\theta^9$, so that

$$\left(y(\theta)^3\right)^{(k)}\Big|_{\theta=0} = 0 \qquad \text{for} \quad k = 0, 1, \ldots, 8 .$$

Hence

$$y^{(5)} = -\left(y^3\right)^{(3)} - \cos\theta \qquad \Rightarrow \qquad y^{(5)}(0) = -\left(y^3\right)^{(3)}\Big|_{\theta=0} - \cos 0 = -1 ,$$

$$y^{(6)} = -\left(y^3\right)^{(4)} + \sin\theta \qquad \Rightarrow \qquad y^{(4)}(0) = -\left(y^3\right)^{(4)}\Big|_{\theta=0} + \sin 0 = 0 ,$$

$$y^{(7)} = -\left(y^3\right)^{(5)} + \cos\theta \qquad \Rightarrow \qquad y^{(7)}(0) = -\left(y^3\right)^{(5)}\Big|_{\theta=0} + \cos 0 = 1 .$$

Thus, the first three nonzero terms of the Taylor approximations are

$$\frac{1}{3!}\theta^3 - \frac{1}{5!}\theta^5 + \frac{1}{7!}\theta^7 + \cdots = \frac{1}{6}\theta^3 - \frac{1}{120}\theta^5 + \frac{1}{5040}\theta^7 + \cdots .$$

3. We will use the ratio test given in Theorem 2 on page 454 of the text to find the radius of convergence for this power series. Since $a_n = \dfrac{n^2}{2^n}$, we see that

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^2}{2^{n+1}}}{\dfrac{n^2}{2^n}} = \frac{(n+1)^2}{2n^2}.$$

Therefore, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{2n^2}\right| = \frac{1}{2}\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\right| = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2}.$$

Thus, the radius of convergence is $\rho = 2$. Hence, this power series converges absolutely for $|x+2| < 2$. That is, for

$$-2 < x+2 < 2 \qquad \text{or} \qquad -4 < x < 0.$$

We must now check the end points of this interval. We first check the end point $x = -4$ or $x + 2 = -2$ which yields the series

$$\sum_{n=0}^{\infty} \frac{n^2(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n n^2 .$$

This series diverges since the nth term, $a_n = (-1)^n n^2$, does not approach zero as n goes to infinity. (Recall that it is necessary for the nth term of a convergent series to approach zero as n goes to infinity. But this fact in itself does not prove that a series converges.) Next, we check the end point $x = 0$ or $x + 2 = 2$ which yields the series

$$\sum_{n=0}^{\infty} \frac{n^2 2^n}{2^n} = \sum_{n=0}^{\infty} n^2 .$$

Again, as above, this series diverges. Therefore, this power series converges in the open interval $(-4, 0)$ and diverges outside of this interval.

9) Given $f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n \qquad g(x) = \sum_{n=1}^{\infty} 2^{-n} x^{n-1}$

Find $f(x) + g(x)$

First, shift index so powers on x are the same

For $g(x)$ let $k = n-1$

$n = k+1 \qquad \therefore g(x) = \sum_{k=0}^{\infty} 2^{-(k+1)} x^k$

For $f(x)$ let $k = n$

$\therefore f(x) = \sum_{k=0}^{\infty} \frac{1}{k+1} x^k$

$$\therefore f(x) + g(x) = \sum_{n=0}^{\infty} \left[\frac{1}{n+1} + 2^{-(n+1)}\right] x^n$$

15. Zero is an ordinary point for this equation since the functions $p(x)=x-1$ and $q(x)=1$ are both analytic everywhere and, hence, at the point $x=0$. Thus, we can assume that the solution to this linear differential equation has a power series expansion with a positive radius of convergence about the point $x=0$. That is, we assume that

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n \ .$$

In order to solve the differential equation we must find the coefficients a_n. To do this, we must substitute $y(x)$ and its derivatives into the given differential equation. Hence, we must find $y'(x)$ and $y''(x)$. Since $y(x)$ has a power series expansion with a positive radius of convergence about the point $x = 0$, we can find its derivative by differentiating term by term. We can similarly differentiate $y'(x)$ to find $y''(x)$. Thus, we have

$$y'(x) = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow \qquad y''(x) = 2a_2 + 6a_3 x + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \ .$$

By substituting these expressions into the differential equation, we obtain

$$y'' + (x-1)y' + y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + (x-1)\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0 \ .$$

Simplifying yields

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0 \ . \qquad (1)$$

We want to be able to write the left-hand side of this equation as a single power series. This will allow us to find expressions for the coefficient of each power of x. Therefore, we first need to shift the indices in each power series above so that they sum over the same powers of x. Thus, we let $k = n-2$ in the first summation and note that this means that $n = k+2$ and that $k = 0$ when $n = 2$. This yields

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k \ .$$

In the third power series, we let $k = n-1$ which implies that $n = k+1$ and $k = 0$ when $n = 1$. Thus, we see that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k \ .$$

For the second and last power series we need only to replace *n* with *k*. Substituting all of these expressions into their appropriate places in equation (1) above yields

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^k = 0 .$$

Our next step in writing the left-hand side as a single power series is to start all of the summations at the same point. To do this we observe that

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k = (2)(1)a_2x^0 + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k,$$

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k = (1)a_1x^0 + \sum_{k=1}^{\infty}(k+1)a_{k+1}x^k,$$

$$\sum_{k=0}^{\infty}a_kx^k = a_0x^0 + \sum_{k=1}^{\infty}a_kx^k.$$

Thus, all of the summations now start at one. Therefore, we have

$$(2)(1)a_2x^0 + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty}ka_kx^k - (1)a_1x^0$$

$$-\sum_{k=1}^{\infty}(k+1)a_{k+1}x^k + a_0x^0 + \sum_{k=1}^{\infty}a_kx^k = 0$$

$$\Rightarrow \qquad 2a_2 - a_1 + a_0 + \sum_{k=1}^{\infty}\left((k+2)(k+1)a_{k+2}x^k + ka_kx^k - (k+1)a_{k+1}x^k + a_kx^k\right) = 0$$

$$\Rightarrow \qquad 2a_2 - a_1 + a_0 + \sum_{k=1}^{\infty}\left((k+2)(k+1)a_{k+2} + (k+1)a_k - (k+1)a_{k+1}\right)x^k = 0.$$

In order for this power series to equal zero, each coefficient must be zero. Therefore, we obtain

$$2a_2 - a_1 + a_0 = 0 \qquad \Rightarrow \qquad a_2 = \frac{a_1 - a_0}{2},$$

and

$$(k+2)(k+1)a_{k+2} + (k+1)a_k - (k+1)a_{k+1} = 0, \qquad k \geq 1$$

$$\Rightarrow \qquad a_{k+2} = \frac{a_{k+1} - a_k}{k+2}, \qquad k \geq 1,$$

where we have canceled the factor $(k+1)$ from the recurrence relation, the last equation obtained above. *Note that in this recurrence relation we have solved for the coefficient with the largest subscript, namely* a_{k+2}. *Also, note that the first value for* k *in the recurrence relation is the same as the first value for* k *used in the summation notation.* By using the recurrence relation with $k = 1$, we find that

$$a_3 = \frac{a_2 - a_1}{3} = \frac{\frac{a_1 - a_0}{2} - a_1}{3} = \frac{-(a_1 + a_0)}{6},$$

where we have plugged in the expression for a_2 that we found above. By letting $k = 2$ in the recurrence equation, we obtain

$$a_4 = \frac{a_3 - a_2}{4} = \frac{\frac{-(a_1 + a_0)}{6} - \frac{a_1 - a_0}{2}}{4} = \frac{-2a_1 + a_0}{12},$$

where we have plugged in the values for a_2 and a_3 found above. Continuing this process will allow us to find as many coefficients for the power series of the solution to the differential equation as we may want. Notice that the coefficients just found involve only the variables a_0 and a_1. From the recurrence equation, we see that this will be the case for all coefficients of the power series solution. Thus, a_0 and a_1 are arbitrary constants and these variables will be our arbitrary variables in the general solution. Hence, substituting the values for the coefficients that we found above into the solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots,$$

yields the solution

$$y(x) = a_0 + a_1 x + \frac{a_1 - a_0}{2} x^2 + \frac{-(a_1 + a_0)}{6} x^3 + \frac{-2a_1 + a_0}{12} x^4 + \cdots$$

$$= a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \cdots\right) + a_1\left(x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{6} + \cdots\right).$$

21. Since $x = 0$ is an ordinary point for this differential equation, we will assume that the solution has a power series expansion with a positive radius of convergence about the point $x = 0$. Thus, we have

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \Rightarrow \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}.$$

By plugging these expressions into the differential equation, we obtain

$$y'' - xy' + 4y = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \quad \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0.$$

In order for each power series to sum over the same powers of x, we will shift the index in the first summation by letting $k = n - 2$, and we will let $k = n$ in the other two power series. Thus, we have

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 4a_k x^k = 0.$$

Next we want all of the summations to start at the same point. Therefore, we will take the first term in the first and last power series out of the summation sign. This yields

$$(2)(1)a_2x^0 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} ka_kx^k + 4a_0x^0 + \sum_{k=1}^{\infty} 4a_kx^k = 0$$

$$\Rightarrow \qquad 2a_2 + 4a_0 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} ka_kx^k + \sum_{k=1}^{\infty} 4a_kx^k = 0$$

$$\Rightarrow \qquad 2a_2 + 4a_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)a_{k+2} + (-k+4)a_k\right]x^k = 0 .$$

By setting each coefficient of the power series equal to zero, we see that

$$2a_2 + 4a_0 = 0 \qquad \Rightarrow \qquad a_2 = \frac{-4a_0}{2} = -2a_0 ,$$

$$(k+2)(k+1)a_{k+2} + (-k+4)a_k = 0 \qquad \Rightarrow \qquad a_{k+2} = \frac{(k-4)a_k}{(k+2)(k+1)}, \qquad k \geq 1,$$

where we have solved the recurrence equation, the last equation above, for a_{k+2}, the coefficient with the largest subscript. Thus, we have

$$k=1 \qquad \Rightarrow \qquad a_3 = \frac{-3a_1}{3\cdot 2} = \frac{-a_1}{2} ,$$

$$k=2 \qquad \Rightarrow \qquad a_4 = \frac{-2a_2}{4\cdot 3} = \frac{(-2)(-4)a_0}{4\cdot 3\cdot 2} = \frac{a_0}{3},$$

$$k=3 \qquad \Rightarrow \qquad a_5 = \frac{-a_3}{5\cdot 4} = \frac{(-3)(-1)a_1}{5\cdot 4\cdot 3\cdot 2} = \frac{a_1}{40},$$

$$k=4 \qquad \Rightarrow \qquad a_6 = 0 ,$$

$$k=5 \qquad \Rightarrow \qquad a_7 = \frac{a_5}{7\cdot 6} = \frac{(-3)(-1)(1)a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} = \frac{a_1}{560},$$

$$k=6 \qquad \Rightarrow \qquad a_8 = \frac{2a_6}{8\cdot 7} = 0 ,$$

$$k=7 \qquad \Rightarrow \qquad a_9 = \frac{3a_7}{9\cdot 8} = \frac{(-3)(-1)(1)(3)a_1}{9!} ,$$

$$k=8 \qquad \Rightarrow \qquad a_{10} = \frac{4a_8}{10\cdot 9} = 0 ,$$

$$k = 9 \qquad \Rightarrow \qquad a_{11} = \frac{5a_9}{11 \cdot 10} = \frac{(-3)(-1)(1)(3)(5)a_1}{11!}.$$

Now we can see a pattern starting to develop. (*Note that it is easier to determine such a pattern if we consider specific coefficients that have not been multiplied out.*) We first note that a_0 and a_1 can be chosen arbitrarily. Next we notice that the coefficients with even subscripts larger than 4 are zero. We also see that the general formula for a coefficient with an odd subscript is given by

$$a_{2n+1} = \frac{(-3)(-1)(1)\cdots(2n-5)a_1}{(2n+1)!}.$$

Notice that this formula is also valid for a_3 and a_5. Substituting these expressions for the coefficients into the solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots,$$

yields

$$y(x) = a_0 + a_1 x - 2a_0 x^2 - \frac{a_1}{2}x^3 + \frac{a_0}{3}x^4 + \frac{a_1}{40}x^5 + \cdots$$

$$+ \frac{(-3)(-1)(1)\cdots(2n-5)a_1}{(2n+1)!}x^{2n+1} + \cdots$$

$$= a_0\left(1 - 2x^2 + \frac{x^4}{3}\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{40} + \cdots + \frac{(-3)(-1)(1)\cdots(2n-5)}{(2n+1)!}x^{2n+1} + \cdots\right)$$

$$= a_0\left(1 - 2x^2 + \frac{x^4}{3}\right) + a_1\left[x + \sum_{k=1}^{\infty} \frac{(-3)(-1)(1)\cdots(2k-5)}{(2k+1)!}x^{2k+1}\right].$$