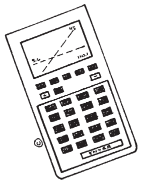


$\mathcal{ME}$ 201 ADVANCED CALCULUS

Assignment 8: *Triple Integrals, Volumes, Centroids & Moments*
March 9, 2018

1. Evaluate the following triple integrals over the given region:

   (a) $\displaystyle\iiint_G xy\,dV$ where G is bounded by $z = \sqrt{1-x^2-y^2},\ z=0$

   ANSWER: 0

   (b) $\displaystyle\iiint_G (x+y+z)\,dV$ where G is bounded by $x=0,\ x=1,\ z=0,$ $y+z=2,\ y=z$

   ANSWER: $11/6$

   (c) $\displaystyle\iiint_G x^2y\,dV$ where G is bounded by $z = \dfrac{x^2}{4}+\dfrac{y^2}{9},\ z=1$

   ANSWER: $48/35$

   (d) $\displaystyle\iiint_G (x^2+y^2+z^2)\,dV$ where G is bounded by $z=\sqrt{1-x^2-y^2},\ z=x^2$

   (set up but do not evaluate)

   (e) $\displaystyle\iiint_G (y+x^2)\,dV$ where G is bounded by $x+z^2=1,\ z=x+1,\ y=1,$ $y=-1$

   ANSWER: $729/70$

2. Find the volume bounded by the following surfaces:

   (a) $x=z^2,\ z=x^2,\ y=0,\ y=2$ ANSWER: $2/3$

   (b) $y=x^2-1,\ y=1-x^2,\ x+z=1,\ z=0$ ANSWER: $8/3$

3. Find the average value of the function $f(x,y,z)=x^2+y^2+z^2$ over the region bounded by $x=0,\ x=1,\ y+z=2,\ y=2, z=2$

   ANSWER: $13/3$

4. A pyramid has a square base with side lengths b and has height h at its center. Find its volume using a triple integral.

   ANSWER: $hb^2/3$

5. Find the volume bounded by the following surfaces using cylindrical coordinates:

   (a) $z = x^2 + y^2, \ z = 4 - x^2 - y^2$ ANSWER: 4π

   (b) $x + y + z = 2, \ x^2 + y^2 = 1, \ z = 0$ ANSWER: 2π

6. Set up the six triple iterated integrals (do not evaluate the integrals) in polar coordinates for the triple integral of the function $f(x, y, z)$ over the region bounded by the surfaces $z = 1 + x^2 + y^2, \ x^2 + y^2 = 9, \ z = 0$.

7. A casting is in the form of a sphere of radius b with two cylindrical holes of radius $a < b$ such that the axes of the holes pass through the center of the sphere and intersect at right angles. What volume of metal is required for the casting?

   ANSWER: $V = \dfrac{16a^3}{3} + \dfrac{4\pi}{2}\left[2(b^2 - a^2)^{3/2} - b^3\right]$

8. Find the volume bounded by the following surfaces using spherical coordinates:

   (a) $z = \sqrt{x^2 + y^2}, \ z = \sqrt{1 - x^2 - y^2}$ ANSWER: $(2 - \sqrt{2})\pi/3$

   (b) $x^2 + y^2 + z^2 = 1, \ y = x, \ y = 2x, \ z = 0$ in the first octant

   ANSWER: $\dfrac{1}{3}(\tan^{-1} 2 - \pi/4)$

9. The temperature distribution of the region $R(r, \phi, \theta)$ in the range, $a \leq r \leq b$, $0 \leq \phi \leq \pi, \ 0 \leq \theta \leq 2\pi$ is given by:

$$T(r) = T_0 + \frac{S(r^2 - a^2)}{6k} + \frac{Sb^3}{3k}\left(\frac{1}{r} - \frac{1}{a}\right)$$

   where T_0, S, k are constants. Determine the volumetric average temperature

$$\overline{T} = \frac{1}{V}\int\int\int_R T \, dV$$

   where V is the total volume of R. Verify that the triple integral reduces to the single integral

$$\overline{T} = \frac{3}{b^3 - a^3}\int_a^b T(r) r^2 \, dr$$

10. Use Archimedes' principle to determine the density of a spherical ball, with density ρ_b, if it floats half submerged in water with density, ρ_w. What force is required to keep the ball with its center at a depth of one-half the radius of the ball.

   ANSWER: $\rho_b = \dfrac{\rho_w}{2}, \ \ F = \dfrac{11}{24}\pi\rho_w g R^3$

11. Given the intersection of a sphere, $x^2 + y^2 + z^2 = z$ and a cone, $z^2 = x^2 + y^2$:

   (a) Sketch the solid and determine the coordinates of the center of the intersecting plane between the two objects. What is the radius of the sphere?

   ANSWER: $a = 1/2$

   (b) Use triple integrals to find the volume of the solid within the sphere and above the cone.

   ANSWER: $\pi/8$

12. Find the centroid of the $2D$ region bounded by the curves:

   (a) $y = 8 - 2x^2,\ y + x^2 = 4$ ANSWER: $(0,\ 24/5)$

   (b) $x = 4y - 4y^2,\ x = y + 3,\ y = 1,\ y = 0$ ANSWER: $(177/85,\ 9/17)$

   (c) $y = \sqrt{2 - x},\ 15y = x^2 - 4$ ANSWER: $(-61/28,\ 807/700)$

13. Find the second moment of inertia of the $2D$ region $y = x,\ y = 2x + 4,\ y = 0$ about the $x-$axis.

   ANSWER: $32/3$

14. Find the centroid of the $3D$ region bounded by the surfaces:

   (a) $y = 4 - x^2,\ y = z,\ z = 0$ ANSWER: $(0,\ 16/7,\ 8/7)$

   (b) $y = x^3,\ x = y^2,\ z = 1 + x^2 + y^2,\ z = -x^2 - y^2$

   ANSWER: $\left(\dfrac{6772}{11847},\ \dfrac{7300}{14001},\ \dfrac{1}{2}\right)$

15. Find the second moment of inertia of the $3D$ region $y + z = 2,\ x + z = 2,\ x = 0,$ $y = 0,\ z = 0$ about the $z-$axis where the material density is given as ρ.

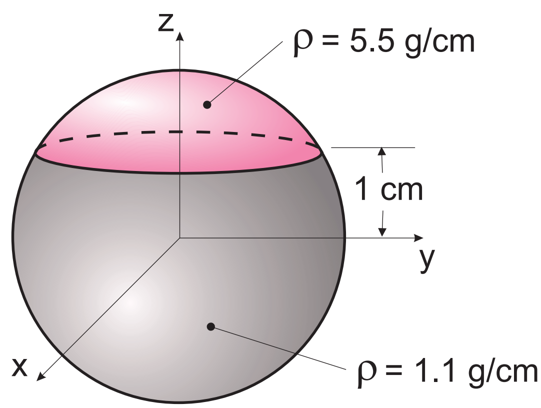
   ANSWER: $64\rho/15$

16. Find the center of mass of a uniform solid in the first octant bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

   ANSWER: $\left(\dfrac{3a}{8}, \dfrac{3b}{8}, \dfrac{3c}{8}\right)$

17. A novelty golf ball, the *Unputtaball*, is described as looking like a real golf ball but when it rolls it wobbles and jumps in all directions, making it impossible to putt - a good prank for your fellow golfers. This $\boldsymbol{4\ cm}$ diameter ball is constructed using a two part rubber core, where each part has a different density. Neglecting the thin coating on the outside of the ball, calculate the center of mass, $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{z}})$, using the coordinate system given in the figure. Hint: perform the volume integral in cylindrical coordinates.



ANSWER: $\left(\mathbf{0}, \mathbf{0}, \dfrac{\mathbf{27}}{\mathbf{52}}\right)$