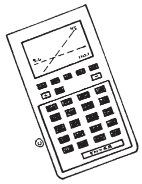

$\mathcal{ME}$ 201 ADVANCED CALCULUS

Assignment 10: *Line Integrals, Conservative Force Fields Scalar Potential Functions & Green's Theorem*

March 23, 2018

1. Evaluate the line integrals of the following scalar functions:

   (a) $\oint_{CCW} (x^2 + y^2)\, ds$ once around the square C in the $xy-$plane with vertices $(-1,-1),\ (-1,1),\ (1,-1),\ (1,1)$.

   ANSWER: $32/3$

   (b) $\int_C x^2yz\, ds$ where C is the curve $z = x+y,\ x+y+z = 1$ from $(1,-1/2,1/2)$ to $(-3,7/2,1/2)$.

   ANSWER: $37\sqrt{2}/3$

   (c) $\int_C (x+y)z\, ds$ where C is the curve $y = x, z = 1 + y^4$ from $(-1,-1,2)$ to $(1,1,2)$.

   (set up the integral but do not evaluate)

2. Find the average value of the function $f(x,y,z) = xyz$ along the curve $C : z = x^2,\ y = x^2$ from $(0,0,0)$ to $(1,1,1)$.

   ANSWER: 0.2417

3. Ice tends to form on the wings of aircraft when air conditions are such that the average air temperature, $\overline{T}_c$, over the flight drops below $0\ °C$ and the relative humidity is 100%. On a particular day of 100% relative humidity, the air temperature distribution is defined in terms of position:

$$T(x,y,z) = -10 + 2\sqrt{x^2 + y^2 + z}\ \ [°C]$$

   as the plane follows the path given by:

$$C: \ \ z = x^2, \quad y = 2x$$

   from $(0,0,0)$ to $(3,6,9)$. Determine the average temperature $\overline{T}_c$. Will ice form on the wings?

4. Evaluate the line integrals of the following vector functions:

   (a) $\int_C x\,dx + yz\,dy + x^2\,dz$ where C is the curve $y = x,\ z = x^2$ from $(-1,-1,1)$ to $(2,2,4)$.

   ANSWER: $51/4$

   (b) $\oint_{CCW} x^2y\,dx\ +\ (x-y)\,dy$ once counterclockwise around the region bounded by the curves $x = 1 - y^2,\ y = x + 1$.

   ANSWER: $-99/140$

5. Find the work done by the given force on a particle as it moves along the given path:

   (a) $\vec{F} = (x^2y)\hat{\imath} + x\hat{\jmath}$ along a straight line joining points $(1,0)$ and $(6,5)$.

   ANSWER: $3235/12$

   (b) $\vec{F} = x\hat{\imath} + y\hat{\jmath}$ once counterclockwise around $b^2x^2 + a^2y^2 = b^2a^2,\ z = 0$.

   ANSWER: 0

6. For each of the following, show that the line integral is independent of path and evaluate it:

   (a) $\int_C 3x^2yz\,dx\ +\ x^3z\,dy\ +\ (x^3y - 4z)\,dz$ where C is the curve $y = x,\ x^2 + y^2 + z^2 = 3$ from $(-1,-1,1)$ to $(1,1,-1)$.

   ANSWER: -2

   (b) $\oint_{CW} y\cos x\,dx\ +\ \sin x\,dy$ once clockwise around the circle $x^2 + y^2 - 2x + 4y = 7,\ z = 0$.

   ANSWER: 0

   (c) $\int_C 3x^2y^3\,dx\ +\ 3x^3y^2\,dy$ where C is the curve $y = e^x$ from $(0,1)$ to $(1,e)$.

   ANSWER: e^3

   (d) $\oint_{CW} y(\tan x + x\sec^2 x)\,dx\ +\ x\tan x\,dy\ +\ dz$ once clockwise around the circle $x^2 + y^2 = 1,\ z = 0$.

   ANSWER: 0

7. Given a scalar potential function $\phi = 3x^2y - y^4 + x^3$ and that $\vec{F} = \nabla\phi$ demonstrate independence of path by calculating the work required to move between $A(0,0)$ and $B(2,4)$ along any path of your choosing.

8. Determine if the force field is conservative and if it is, find its potential function:

   (a) $\vec{F} = mx\hat{\imath} \ + \ xy\hat{\jmath}$ (m is a constant) ANSWER: not conservative

   (b) $\vec{F} = -mg\hat{k}$ (m and g are constants) ANSWER: conservative, $U = -mgz$

9. One end of a spring with unstretched length L is fixed at the origin in space. If the other end is at point (x, y, z) (all coordinates in meters) what is the force exerted by the spring? Is the force conservative? ANSWER: conservative

10. Find the work done by the force field:

$$\vec{F} = \left(-\frac{y}{z}\sin x\right)\hat{\imath} \ + \ \left(\frac{1}{z}\cos x\right)\hat{\jmath} \ - \ \left(\frac{y}{z^2}\cos x\right)\hat{k}$$

if it acts on a particle that moves:

   (a) from $(\sqrt{2}, \sqrt{2}, 2\pi)$ to $(\sqrt{2}, \sqrt{2}, 4\pi)$ along $x = 2\cos t,\ y = 2\sin t,$
   $z = t + \dfrac{7\pi}{4}, \dfrac{\pi}{4} \le t \le \dfrac{9\pi}{4}$
   ANSWER: -0.0175

   (b) from $(\sqrt{2}, \sqrt{2}, 2\pi)$ to $(\sqrt{2}, \sqrt{2}, 4\pi)$ along a straight line
   ANSWER: -0.0175

   (c) once around the circle $x^2 + y^2 = 4,\ z = 2\pi$
   ANSWER: 0

11. Use Green's theorem to evaluate the following line integrals:

   (a) $\oint_{CCW} xy^3\,dx \ + \ x^2\,dy$ where C encloses the region bounded by
   $x = \sqrt{1+y^2},\ x = 2.$
   ANSWER: $2\sqrt{3}/5$

   (b) $\oint_{CCW} 2\tan^{-1}(y/x)\,dx + \ln(x^2+y^2)\,dy$ where C is the circle
   $(x-4)^2 + (y-1)^2 = 2$
   ANSWER: 0

   (c) $\oint_{CCW} (x^3+y^3)\,dx \ + \ (x^3-y^3)\,dy$ where C is the curve $2|x| + |y| = 1.$
   ANSWER: $-3/8$