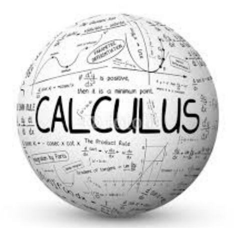
Vector Calculus

Vector Fields



Reading
Trim 14.1 $\longrightarrow$ *Vector Fields*

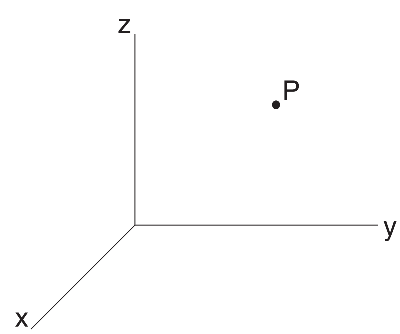
Assignment
web page $\longrightarrow$ *assignment #9*

Chapter 14 will examine a vector field.

For example, if we examine the temperature conditions in a room, for every point P in the room, we can assign an air temperature, T, where

$$T = f(x, y, z)$$

This is a scalar function or scalar field.

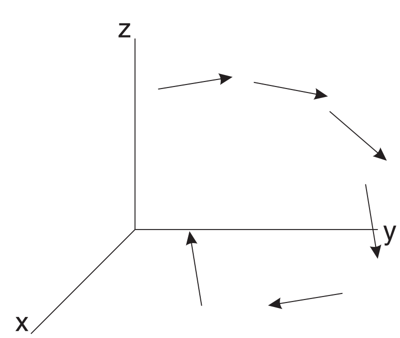


However, suppose air is moving around in the room and at every point P, we can assign an air velocity vector, $\vec{V}(x, y, z)$, where

$$\vec{V}(x, y, z) = \hat{i}\, u(x, y, z) + \hat{j}\, v(x, y, z) + \hat{k}\, w(x, y, z)$$

where the right side of the above equation consists of x, y, z components at a point, with each component being a function of (x, y, z). To describe $\vec{V}$ in the room, we need to keep track of the 3 scalar function, u, v, w at each $f(x, y, z)$.

$\vec{V}(x, y, z)$ is a vector function or a vector field.



Gradient, Divergence and Curl Operations

The basic operator is the del operator given as

2D

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$$

3D

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The del operator operates on a scalar function, such as $T = f(x, y, z)$ or on a vector function, such as $\vec{F} = \hat{i}P + \hat{j}Q + \hat{k}R$ (force) or $\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$ (velocity).

We will examine three operations is more detail

1. **gradient:** ∇ operating on a scalar function

   Examples include the temperature in a 2D plate, $T = f(x, y)$

   $$\nabla T = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}$$

   or for a 3D volume, such as a room

   $$\nabla T = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

   The physical meaning was given in Chapter 13. ∇T is a vector perpendicular to the T contours that points "uphill" on the contour plot or level surface plot.

2. **divergence:** ∇ dotted with a vector function $\rightarrow \quad \nabla \cdot \vec{V}$

   We can define the velocity in a 3D room as

   $$\vec{V}(x, y, z) = \hat{i}u(x, y, z) + \hat{j}v(x, y, z) + \hat{k}u(x, y, z)$$

   $$\text{DIVERGENCE of } \vec{V} = \text{div } \vec{V} = \nabla \cdot \vec{V}$$

   $$\nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}u + \hat{j}v + \hat{k}w\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

   The DIVERGENCE of a vector is a scalar.

3. **curl:** the vector product of the del operator and a vector, $\nabla \times \vec{V}$, produces a vector

$$\text{curl of } \vec{V} = \text{curl } \vec{V} = \nabla \times \vec{V}$$

$$\begin{aligned}
\nabla \times \vec{V} &= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z} \right) \times \left(\hat{i}u + \hat{j}v + \hat{k}w \right) \\
&\equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\
&= \hat{i} \left(\underbrace{\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}}_{V_1} \right) + \hat{j} \left(\underbrace{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}}_{V_2} \right) + \hat{k} \left(\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{V_3} \right) \\
&= \hat{i}\, V_1 + \hat{j}\, V_2 + \hat{k}\, V_3
\end{aligned}$$

The **3D** case gives

$$\boxed{\text{curl } \vec{V} = \hat{i} \left(\underbrace{\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}}_{V_1} \right) + \hat{j} \left(\underbrace{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}}_{V_2} \right) + \hat{k} \left(\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{V_3} \right)}$$

The **2D** case gives

$$\boxed{\text{curl } \vec{V} = \hat{k} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)}$$

4. **Laplacian:** $(\nabla \cdot \nabla)$ operation.

   A primary example of the Laplacian operator is in determining the conduction of heat in a solid. Given a 3D temperature field $T(x, y, z)$, the Laplacian is

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} = 0}$$

2D example

Consider a 2D heat flow field, $\vec{q}(x, y)$.

Look at a small differential element, $\Delta x \Delta y$. In steady state, heat flows in is equivalent to heat flow out. Therefore

$$\nabla \cdot \vec{q} = 0 \qquad (1)$$

But $\vec{q}$ is related to temperature by Fourier's law

$$\vec{q} = -k \nabla T \qquad (2)$$

Combining (1) and (2)

$$\nabla \cdot (-k \nabla T) = 0 \rightarrow \nabla^2 T = 0$$

This is Laplace's equation in 2D, which gives the steady state temperature field.

Example 4.1

Show for the 3D case $f(x, y, z)$ that **curl grad** $f = 0$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

holds for any function $f(x, y, z)$, for instance

$$f(x, y, z) = x^2 + y^2 + y \sin x + z^2$$

Conservative Force Fields and the $\mathrm{curl}\ \mathrm{grad}\ f = 0$ identity

Suppose we have a conservative field $\vec{F}(x, y, z)$. We know $\vec{F}$ is irrotational, i.e.

$$\nabla \times \vec{F} = 0 \qquad (1)$$

(zero work in a closed path in the field)

But the identity says

$$\nabla \times (\nabla\phi) = 0 \qquad (2)$$

always holds when $\phi(x, y, z)$ is a scalar function.

Comparing (1) and (2) $\rightarrow$ for a Conservative Force Field, we can always find a scalar function $\phi(x, y, z)$ such that

$$\vec{F} = \nabla\phi$$

where ϕ is called the scalar potential function.

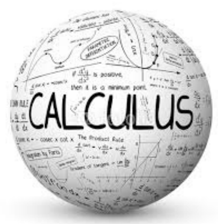
Sometimes, for convenience, we introduce a negative sign

$$\vec{F} = -\nabla u$$

The following statements are equivalent

$\vec{F}$ is a conservative force field	$\Longleftrightarrow$	net work when a particle moves through $\vec{F}$ around a closed path in space is zero
	$\Longleftrightarrow$	$\nabla \times \vec{F} = 0$ irrotational
	$\Longleftrightarrow$	a scalar function $\phi(x, y, z)$ can be found such that $\vec{F} = \nabla\phi$ or a function $u(x, y, z)$ can be found such that $\vec{F} = -\nabla u$

Line Integrals of Scalar Functions

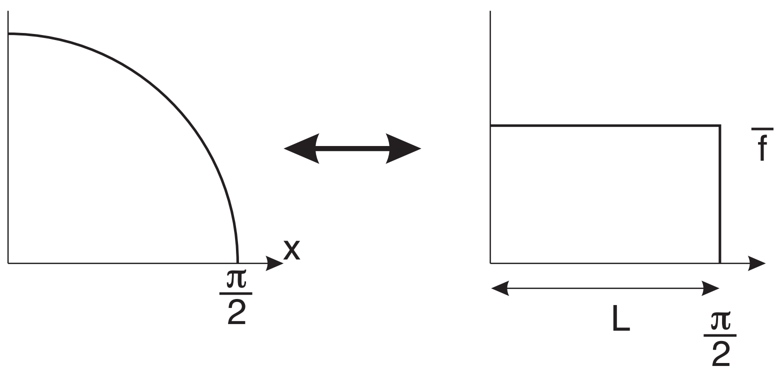


Reading
Trim 14.2 $\longrightarrow$ *Line Integrals*

Assignment
web page $\longrightarrow$ *assignment #10*

One place that line integrals often come up is in the computation of averages of a function.

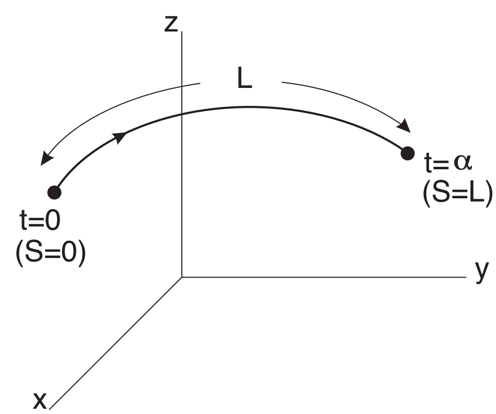
2D Case



$$\overline{f} = \frac{1}{L}\int_{x=0}^{L} f(x)\,dx$$

3D Case

The temperature in a room is given by $T = f(x, y, z)$. A curve C in the room is given by $x(t), y(t)$ and $z(t)$. If we measure temperature along curve C, what is the average temperature $\overline{T}$?



$$\overline{T} = \frac{1}{L}\underbrace{\overbrace{\int_C f(x,y,z)}^{value\ of\ f\ along\ C}\ \overbrace{dS}^{arc\ length\ of\ C}}_{line\ integral\ along\ C}$$

Calculation of the line integral along C

$$\int_{t=0}^{\alpha} \underbrace{f[x(t), y(t), z(t)]}_{f\ values\ along\ C}\ \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}_{dS\ along\ C}\ dt$$

Example 4.2

Given a 3D temperature field

$$T = f(x, y, z) = 8x + 6xy + 30z$$

find the average temperature, $\overline{T}$ along a line from $(0, 0, 0)$ to $(1, 1, 1)$.

1. if we have an explicit equation for a planar curve

   $$C: \qquad y = g(x)$$

   we can reduce

   $$\int_C f\,dS \quad \text{to} \quad \int \text{fnc of } x \;\; dx \quad \text{or} \quad \int \text{fnc of } y \;\; dy$$

   we do not have to always use the parametric equations.

2. value of $\int_C f\ dS$ depends on

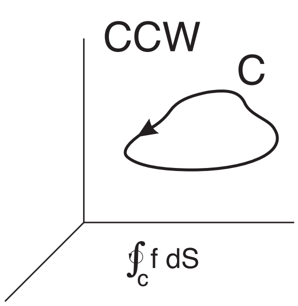
   (i) function f

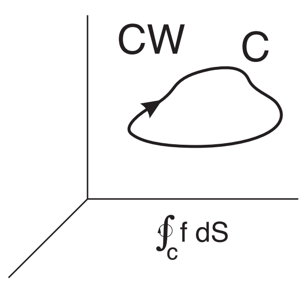
   (ii) curve C in space

   (iii) direction of travel

   $$\int_A^B f\ dS = -\int_B^A f\ dS$$

3. notation - sometimes C is a closed loop in space





- evaluate once around the loop CCW or CW
- evaluation method is the same as the example

Example: 4.3a

Suppose the temperature near the floor of a room (say at $z = 1$) is described by

$$T = f(x, y) = 20 - \frac{x^2 + y^2}{3} \qquad \text{where} \quad -5 \leq x \leq 5$$

$$-4 \leq y \leq 4$$

What is the average temperature along the straight line path from $A(0, 0)$ to $B(4, 3)$.

Example: 4.3b

What is the average room temperature along the walls of the room?

$$\overline{T} = \frac{\oint_C f(x, y) dS}{\oint_C dS}$$

where the closed curve C is defined in 4 sections

C_1	$y = -4$	$x = t$	$-5 \leq t \leq 5$
C_2	$x = 5$	$y = t$	$-4 \leq t \leq 4$
C_3	$y = 4$	$x = 5 - t$	$0 \leq t \leq 10$
C_4	$x = -5$	$y = 8 - t$	$0 \leq t \leq 8$

Example: 4.3c

What is the average temperature around a closed circular path $\rightarrow \quad x^2 + y^2 = 9$?

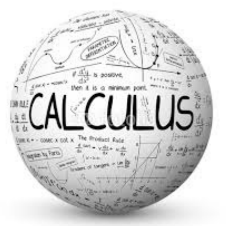
where C is a closed circular path

$$\overline{T} = \frac{\oint_C f(x, y) dS}{\oint_C dS} \qquad \text{where} \quad x(t) = 3 \cos t$$

$$y(t) = 3 \sin t$$

for $0 \leq t \leq 2\pi$.

Line Integrals of Vector Functions



Reading

Trim 14.3 $\longrightarrow$ *Line Integrals Involving Vector Functions*

14.4 $\longrightarrow$ *Independence of Path*

Assignment

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Let's examine work or energy in a force field.

$$W = \vec{F} \cdot \vec{d}$$

Now consider a particle moving along a curve C is a 3D force field.

$$W = \int_C \underbrace{\vec{F}(x, y, z)}_{\textit{force value evaluated along } C} \cdot \underbrace{d\vec{r}}_{\textit{displacement along } C} \qquad (1)$$

This is a line integral of the vector $\vec{F}$ along the curve C in 3D space.

In component form:

$$\vec{F} = \hat{\imath}P + \hat{\jmath}Q + \hat{k}R$$

$$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$$

$$W = \int_C Pdx + Qdy + Rdz \qquad (2)$$

Equations (1) and (2) are equivalent.

Use the equation of curve C (either in an explicit form, i.e. $y = f(x)$ etc., or in a parametric form) to reduce (2) to $\int_{t=0}^{\alpha} g(t)dt$ or $\int_{x=n}^{\beta} h(x)dx$ etc.

Example 4.4

Given a force field in 3D:

$$\vec{F} = \hat{\imath}(3x^2 - 6yx) + \hat{\jmath}(2y + 3xz) + \hat{k}(1 - 4xyz^2)$$

What is the work done by $\vec{F}$ on a particle (i.e. energy added to the particle) if it moves in a straight line from $(0, 0, 0)$ to $(1, 1, 1)$ through the force field.

Notes

1. If we have an explicit equation for the curve, we can sometimes reduce to a form $\int(\textbf{fnc of } x)\, dx$ etc. There is no need for parametric equations.

2. The work term W can be either +'ve or -'ve.
   In a +'ve form, $\int \vec{F}$ and $\int d\vec{r}$, are in the same direction, where the work done by the force energy is added to the object by $\vec{F}$.
   In the -'ve form, $\vec{F}$ opposes the displacement. The energy is removed from the object.
   +'ve $W = \int_C \vec{F} \cdot d\vec{r}$

3. closed path notation

$$
\begin{aligned}
W &= \oint_{CCW} \vec{F} \cdot d\vec{r} \quad \textbf{once CCW around loop} \\
W &= \oint_{CW} \vec{F} \cdot d\vec{r} \quad \textbf{once CW around loop}
\end{aligned}
$$

4. A special case is the conservative force field

$$\oint \vec{F} \cdot d\vec{r} = 0$$

   We know that $\nabla \times \vec{F} = \mathbf{0}$. There is no work in a closed loop.
   We also know that $\nabla\phi = F$, which is integrated gives

$$\int_C \vec{F} \cdot d\vec{r} = \phi_1 - \phi_2$$

   Therefore for a conservative force field, the work is a function of the end points not the path.
   This is the same for any C connecting the same 2 end points.
   W is always the same number if $\vec{F}$ is conservative.

Example: 4.5a

The gravitational force on a mass, m, due to mass, M, at the origin is

$$\vec{F} = -G\frac{Mm\vec{r}}{|\vec{r}|^3} = -K\frac{\vec{r}}{|\vec{r}|^3} \qquad \text{where} \quad K = GMm$$

The vector field is given by:

$$\vec{F}(x,y,z) = \hat{i}P + \hat{j}Q + \hat{k}R \qquad \text{where} \quad P = -\frac{Kx}{(x^2+y^2+z^2)^{3/2}}$$

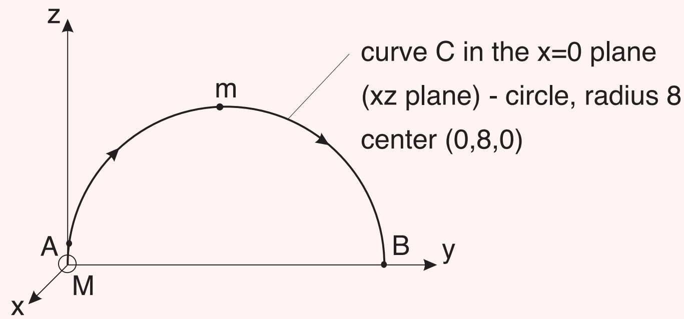
$$Q = -\frac{Ky}{(x^2+y^2+z^2)^{3/2}}$$

$$R = -\frac{Kz}{(x^2+y^2+z^2)^{3/2}}$$

Compute the work, W, if the mass, m moves from A to B along a semi-circular path in the (y,z) plane:

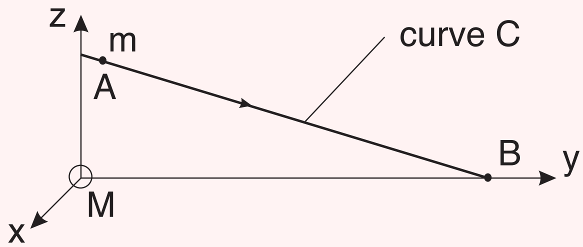
$$y^2 + z^2 = 16y \qquad \text{or} \qquad z = \sqrt{16y - y^2} \quad \text{and } x = 0$$

From $A(0, 0.1, 1.261)$ to $B(0, 16, 0)$

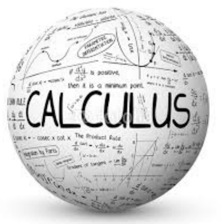


Example: 4.5b

Find the work to move through the same field, but following a straight line path from $A(0, 0.1, 1.261)$ to $B(0, 16, 0)$.



Conservative Force Fields



Reading
Trim 14.5 ⟶ *Energy and Conservative Force Fields*

Assignment
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Given a flow field in 3D space

$$\vec{F} = \hat{i}\,\underbrace{(2xz^3 + 6y)}_{P} + \hat{j}\,\underbrace{(6x - 2yz)}_{Q} + \hat{k}\,\underbrace{(3x^2z^2 - y^2)}_{R}$$

Part a: Is the force field, $\vec{F}$, conservative?

Check to see if $\nabla \times \vec{F} = 0$ (i.e. irrotational $\vec{F}$?)

Part b: Compute the work done on an object if it goes around a CCW circular path of radius 1 for center point $P(2, 0, 3)$ with form

Part c: Find the scalar potential function $\phi(x, y, z)$

$$\vec{F} = \nabla\phi$$

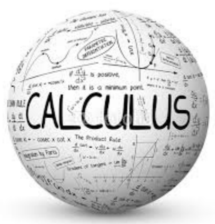
Part d: Use ϕ to verify part b).

$$W = \oint \vec{F} \cdot d\vec{r} = \oint \nabla\phi \cdot d\vec{r}$$

Part e:

Find the work done if the object moves along C from $A(0, 0, 0)$ to $B(3, 4, -2)$ within $\vec{F}$.

Surface Integrals of Scalar Functions



Reading
Trim 14.7 $\longrightarrow$ *Surface Integrals*

Assignment
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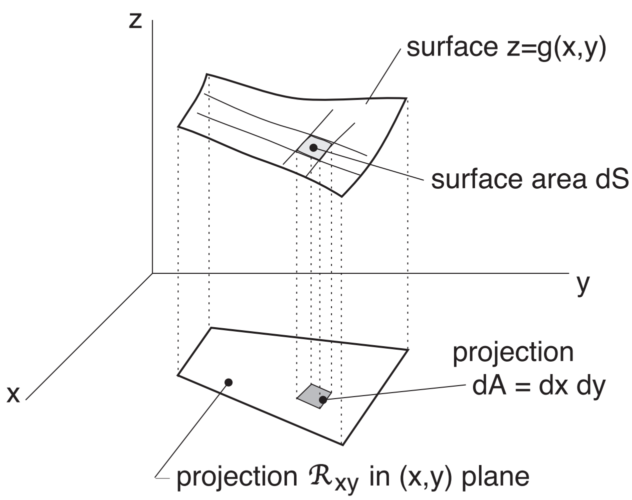


surface area of $z = g(x, y)$

$$S = \int\int_{\mathcal{R}_{xy}} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dxdy$$

Now suppose the surface $z = g(x, y)$ is within a 3D temperature field

$$T = f(x, y, z)$$

What is the average temperature measured over the surface $z = g(x, y)$?

Add up T for each dS area element and divide by the total area of S to get the average.

$$\overline{T} = \frac{\int\int_S f(x, y, z)dS}{Area\ of\ S}$$

The numerator is called the surface integral of $f(x, y, z)$ over the surface S (i.e. $z = g(x, y)$).

To evaluate

$$\int\int_S f dS = \int\int_{\mathcal{R}_{xy}} \underbrace{f[x, y, g(x, y)]}_{T\ values\ on\ the\ surface} \underbrace{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dxdy}_{dS\ area\ element}$$

This becomes

$$\int\int_{\mathcal{R}_{xy}} F(x, y)dxdy$$

Notes

1. sometimes it is easier if we switch to polar coordinates

$$\int\int_{\mathcal{R}_{xy}} F(x,y)dxdy \qquad \rightarrow \qquad \int\int_{\mathcal{R}_{xy}} H(r,\theta)rdrd\theta$$

   where $x = r\cos\theta$ and $y = r\sin\theta$.

2. notation = we have a closed surface in space, i.e. a sphere surface

$$g(x,y) \rightarrow z = \sqrt{a^2 - x^2 - y^2}$$

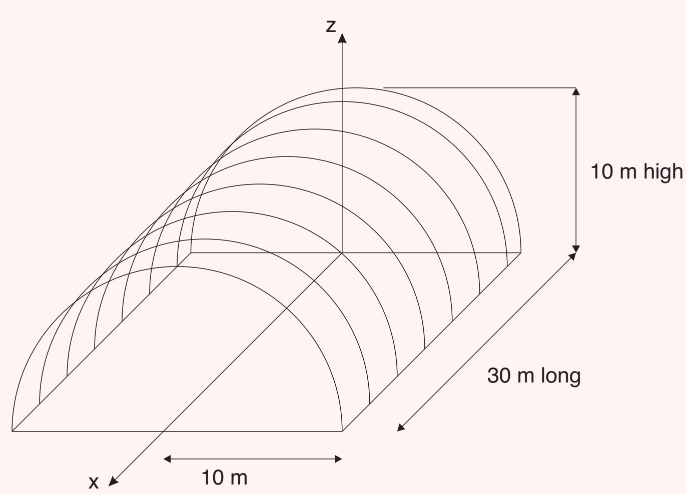
$$\oint\oint_S f(x,y,z)dS$$

Example: 4.6

Suppose the temperature variation (same for all (x,y)) in the atmosphere near the ground is

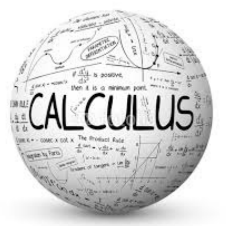
$$T(z) = 40 - \frac{z^2}{5}$$

where T is in $^\circ C$ and z is in m. Look at a cylindrical building roof as follows:



What is the air temperature in contact with the roof?

Surface Integrals of Vector Functions



Reading
Trim 14.8 ⟶ *Surface Integrals Involving Vector Fields*

Assignment
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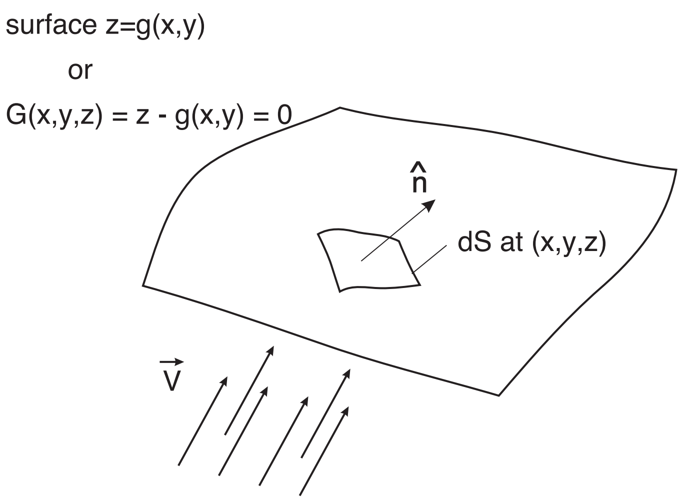
Given a full 3D velocity field, $\vec{V}(x, y, z)$ in space (i.e. air flow in a room).

Given some surface $z = g(x, y)$ within the flow → calculate the flow rate Q (m^3/s) crossing the surface S, we can write $G = z - g(x, y)$, where G is a constant since the surface is a level surface or a contour.



The basic idea is to consider the element dS of the surface at some arbitrary (x, y, z). Then compute the unit normal vector to dS

$$\hat{n} = (\pm)\frac{\nabla G}{|\nabla G|}$$

This will vary over S. The $(\pm)$ will be controlled by the direction of the flow.

The flow across dS is $\underbrace{(\vec{V} \cdot \hat{n})}_{component\ normal\ to\ surface} dS$.

Add up over all dS elements to get the total flow across the surface

$$Q = \int\int_S \vec{V} \cdot \hat{n}\, dS$$

This is called the surface integral of the vector field $\vec{V}$ over the surface S defined by $z = g(x, y)$.

The actual evaluation is similar to the last example.

- $\vec{V} \cdot \hat{n}$ will end up giving some integrand function f
- project dS onto (x, y) plane

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dxdy$$

$$Q = \int\int_{\mathcal{R}_{x,y}} f(x, y) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dxdy$$

- then proceed as before.

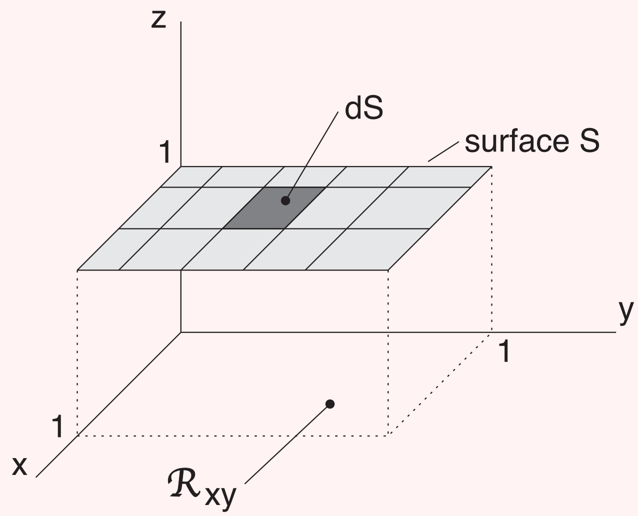
Example: 4.7

Given a velocity field in 3D space

$$\vec{V} = \hat{\imath}(2x + z) + \hat{\jmath}(x^2 y) + \hat{k}(xz) \qquad \text{find}$$

a) the flow rate Q (m^3/s) across the surface $z = 1$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$ in the +'ve z direction

b) the average velocity across the surface



Integral Theorems Involving Vector Functions

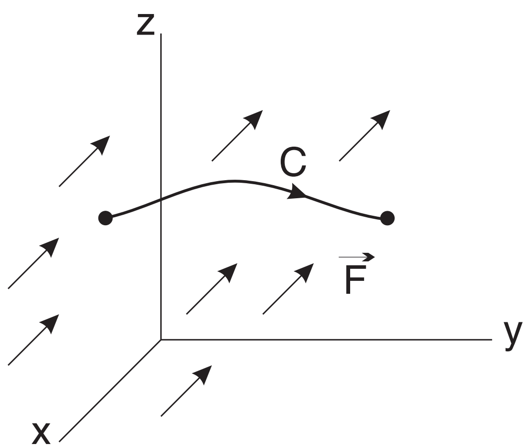
We will examine vector functions in 3D:

Force Field: $\vec{F}(x, y, z) = \hat{i}P + \hat{j}Q + \hat{k}R$

Vector Field: $\vec{V}(x, y, z) = \hat{i}u + \hat{j}v + \hat{k}w$

We have defined 2 types of integrals for such functions.

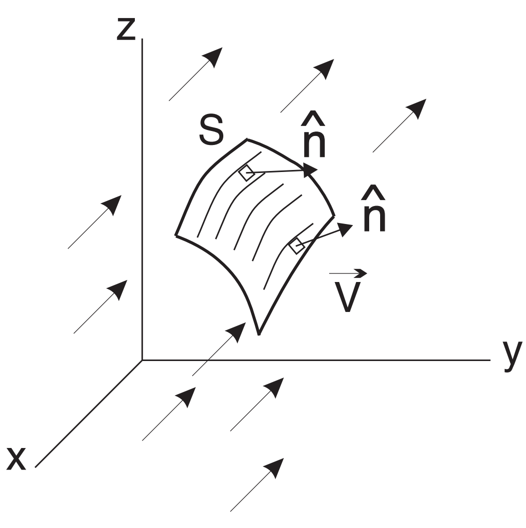
Line Integrals



$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz$$

This can be interpreted as work done by $\vec{F}$ field on an object that moves along C within the field.

Surface Integrals

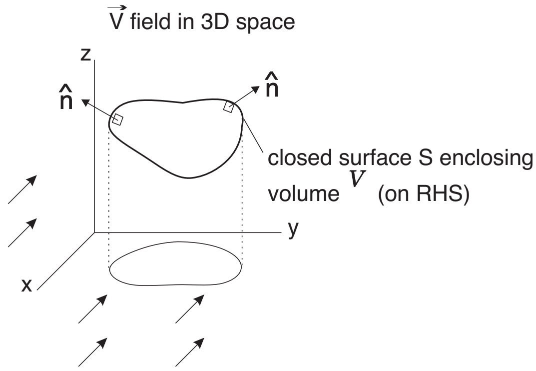


$$\int\int_S \vec{V} \cdot \hat{n}\, dS$$

This can be interpreted as the flow across surface S in the $\hat{n}$ direction due to the $\vec{V}$ field.

There are three theorems which state identities involving these types of integrals.

1. **Divergence Theorem**



Also called Green's theorem in space - this is the 2nd vector form of Green's theorem.

$$\oint\oint_S \vec{V} \cdot \vec{n}\, dS = \int\int\int_{\mathcal{V}} (\nabla \cdot \vec{V}) d\mathcal{V}$$

where

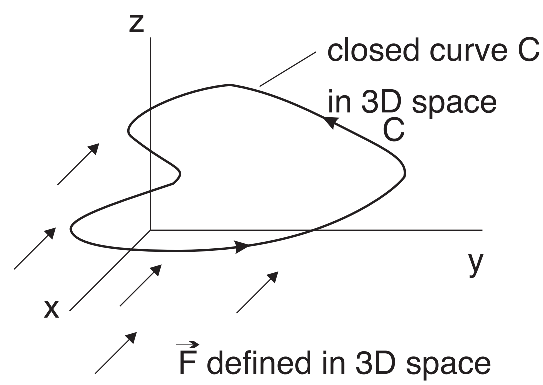
$$\vec{V} = \text{velocity field}$$
$$\mathcal{V} = \text{volume}$$

2. **Stokes Theorem**

Also called 1st vector from of Green's theorem.

$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

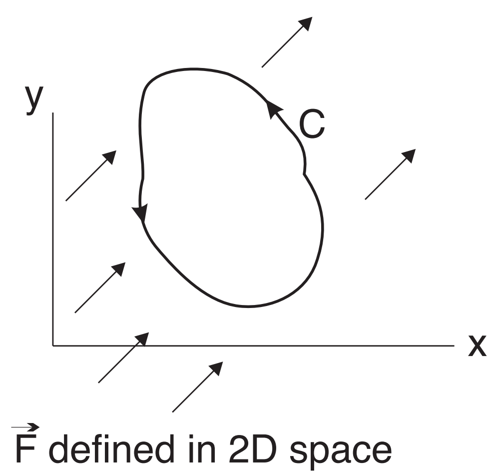
where the surface S is any surface in 3D with C as a boundary.



3. **Greens Theorem**

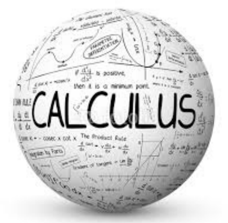
This is essentially a 2D statement of Stoke's theorem, where in 2D

$$\vec{F} = \hat{i}P + \hat{j}Q$$
$$\nabla \times \vec{F} = \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$



$$\oint_C Pdx + Qdy = \int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Divergence Theorem



Reading
Trim 14.9 $\longrightarrow$ *The Divergence Theorem*

Assignment
web page $\longrightarrow$ *assignment #11*

Trim in section 14.9 has a detailed proof of the Divergence Theorem. They try to interpret the meaning of

$$\oint \oint_S \vec{V} \cdot \hat{n} dS = \int \int \int_{\mathcal{V}} (\nabla \cdot \vec{V}) d\mathcal{V}$$

This equation applies for any vector function $\vec{V}$, but is used most for velocity fields in fluids. When we consider $\vec{V}$, the theorem concerns net outflow to inflow (m^3/s) for a region in space (like a sphere).

The left side of the equation is a surface integral of V over a closed surface, S in 3-D space with $\hat{n}$ being the outward normal to each dS.

We recall that $\vec{V} \cdot \hat{n} dS$ gives the flow rate (m^3/s). When we add this up over the entire surface (as in the LHS of the equation) we obtain the **net flow rate crossing the closed surface**.

i.e.

$$\text{net outward flow}(m^3/s) - \text{net inward flow}(m^3/s)$$

The right hand side of the equation is a calculation of $\nabla \cdot \vec{V}$ for each differential volume, $d\mathcal{V}$ inside the surface S. We then add them all up.

For the differential volume, $\mathcal{V}$

$$\begin{aligned} \text{inflow } (m^3/s) &= u(x) \cdot area + v(y)(\Delta x)(\Delta z) \\ \text{outflow } (m^3/s) &= u(x + \Delta x) \cdot (\Delta y)(\Delta z) + v(y + \Delta y)(\Delta x)(\Delta z) \end{aligned}$$

The net flow is then

$$\text{outflow} - \text{inflow} = (\Delta x)(\Delta y)(\Delta z) \left[\frac{u(x + \Delta x) - u(x)}{\Delta x} + \frac{v(y + \Delta y) - v(y)}{\Delta y} \right]$$

$$
\begin{aligned}
&= (d\mathcal{V})\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
&= (\nabla \cdot \vec{V})d\mathcal{V}
\end{aligned}
$$

Therefore the RHS $(\nabla \cdot \vec{V})d\mathcal{V}$ gives the net outflow minus inflow for a volume element $d\mathcal{V}$.

The integral $\int\int\int_{\mathcal{V}}$, adds up the differential flow for all volume elements, $d\mathcal{V}$ inside of surface S. There is a cancellation of terms because the outflow from one $\Delta\mathcal{V}$ becomes the inflow to the next volume.

When we sum over all $\Delta\mathcal{V}$, we are left with the difference between the inflow and the outflow at the boundaries of the volume.

$$
\begin{aligned}
\oint\oint_S \vec{V} \cdot \vec{n} dS \quad &= \quad \int\int\int_{\mathcal{V}} (\nabla \cdot \vec{V})d\mathcal{V} \\
\text{outflow} - \text{inflow}(m^3/s) \quad & \quad \text{triple sum of all outflow} - \text{inflow} \\
\text{across boundary surface} \quad & \quad \text{for } \Delta\mathcal{V} \text{ volumes inside} \\
S \text{ of volume } \mathcal{V} \text{ in 3D space} \quad & \quad \text{the volume } \mathcal{V} \text{ in 3D space}
\end{aligned}
$$

Example: 4.8

Given

$$\vec{V} = \hat{\imath}(1 + x) + \hat{\jmath}(1 + y^2) + \hat{k}(1 + z^3)$$

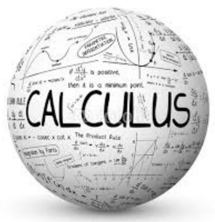
verify the divergence theorem for a cube, where $0 \leq x, y, z \leq 1$ i.e. show that

$$\oint\oint_S \vec{V} \cdot \hat{n} dS = \int\int\int_{\mathcal{V}} (\nabla \cdot \vec{V})d\mathcal{V}$$

where

S = cube surface (closed)
$\mathcal{V}$ = interior volume of the cube

Stoke's Theorem



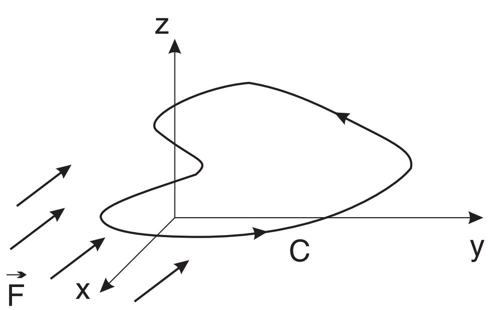
Reading
Trim 14.10 ⟶ *Stoke's Theorem*

Assignment
web page ⟶ *assignment #11*

The formal proof is offered in Trim 14.10.

$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

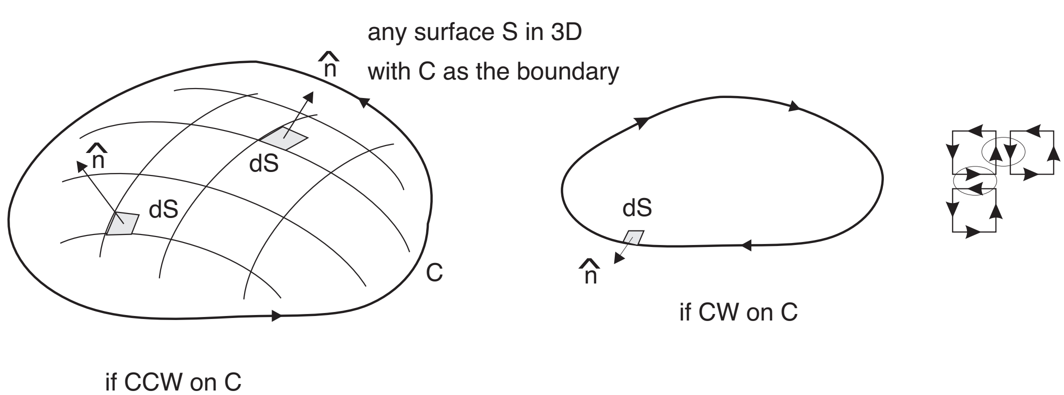
This has a similar meaning to Green's theorem but now in 3D space instead of a plane.



$\vec{F}(x, y, z)$ is a 3D force field. The LHS of the equations is the work done when the object moves once in a CCW direction along the path C in 3D.
The RHS is the work computed over a surface integral $\int\int_S$.

Like Greens theorem, it works because of the interior cancellations of work, when we move around a surface element dS inside C.

Note: the unit normal, $\hat{n}$ for the LHS is based on a right hand rule as follows.



The work on all internal surfaces cancel, leaving only the surface work in the CCW direction.

Example: 4.9

Given: $\vec{F} = \hat{i}x + \hat{j}2z + \hat{k}y$ (a force field in 3D).

The closed path C is given by the intersection of:

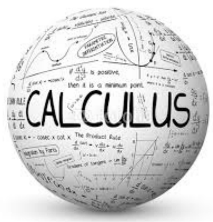
$$\begin{aligned} x^2 + y^2 &= 4 \\ z &= 4 - x - y \end{aligned}$$

The object moves once in a CW direction around C starting at $(2, 0, 2)$.

Verify Stoke' theorem:

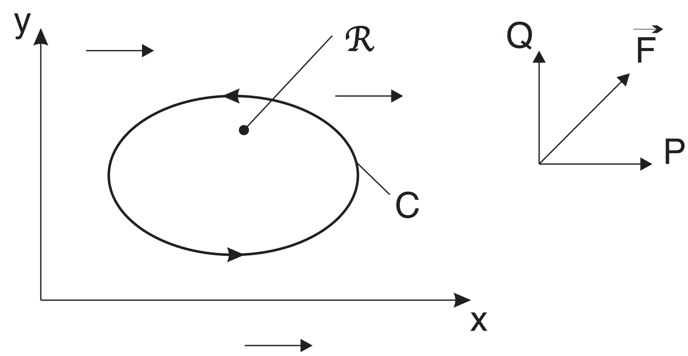
$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

Green's Theorem



Reading
Trim 14.6 $\longrightarrow$ *Green's Theorem*

Assignment
web page $\longrightarrow$ *assignment #10*



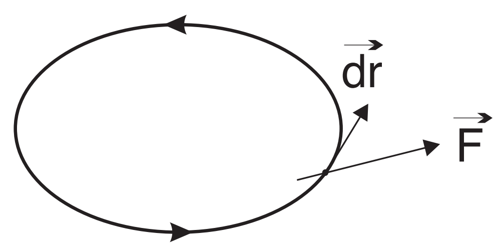
The theorem involves work done on an object by a 2D force field. The 2D force field is given by

$$\boldsymbol{F}(x,y) = \hat{\imath}P(x,y) + \hat{\jmath}Q(x,y)$$

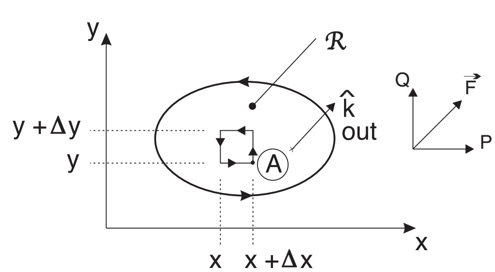
We will examine an object that moves once CCW around a loop in $\vec{F}$. The region inside the loop is defined as $\mathcal{R}$. The normal vector for the region $\mathcal{R}$ is $\hat{k}$ (outwards) to the right for a CCW motion of C. The theorem states

$$\oint_C P\,dx + Q\,dy = \int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

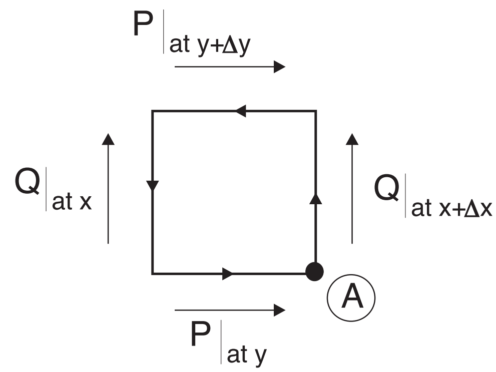
The **left hand side** of the equation, $\oint_C \vec{F} \cdot d\vec{r}$ in 2D is the work done by the field $\vec{F}$ on the object as it moves on C. This consists of force times distance for each $d\vec{r}$ added up over C.



The **right hand side** of the equation, $\int\int_{\mathcal{R}} (\nabla \times \vec{F}) \cdot \hat{k}\,dx\,dy$ in 2D gives the direction of travel on C for the work given by the LHS. Maps movement of an element $\Delta x \Delta y$ as it moves around from A to A.



The force component times the distance for each side gives the work done.

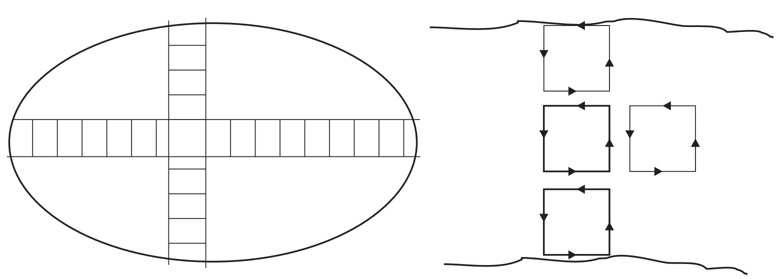


$$\begin{aligned} Work &= Q|_{at\ x+\Delta x}\,\Delta y - P|_{at\ y+\Delta y}\,\Delta x - Q|_{at\ x}\,\Delta y + P|_{at\ y}\,\Delta x \\ &= (\Delta x \Delta y)\left[\frac{Q|_{at\ x+\Delta x} - Q|_{at\ x}}{\Delta x} - \frac{P|_{at\ y+\Delta y} - P|_{at\ y}}{\Delta y}\right] \end{aligned}$$

In the limit, the work done by $\vec{F}$ to move the object CCW around $dxdy$ area is

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Now we can sum up for all $dxdy$ elements inside C.



There are some cancellations (+) (-) for all interior $\Delta x, \Delta y$ paths. Therefore when we do $\oint \oint_{\mathcal{R}}$, we are left with the work terms on the boundaries of $\mathcal{R}$.

$$\oint_C Pdx + Qdy = \int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

work when we	**sum of all work if**
move once CCW around	**move once CCW around all**
boundary curve C	**($dxdy$) area elements of $\mathcal{R}$**
	inside C

Example: 4.10

Given a 2D force field, $\vec{F}(x,y) = \hat{i}(xy^3) + \hat{j}(x^2y)$ and a path C in the field:

Verify Green's theorem

$$\oint_C Pdx + Qdy = \int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

with

$$P = xy^3$$

$$Q = x^2y$$