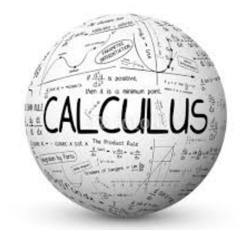
Multiple Integrals

Review of Single Integrals



Reading

Trim 7.1 $\longrightarrow$ *Review Application of Integrals: Area*

7.2 $\longrightarrow$ *Review Application of Integrals: Volumes*

7.3 $\longrightarrow$ *Review Application of Integrals: Lengths of Curves*

Assignment

web page $\longrightarrow$

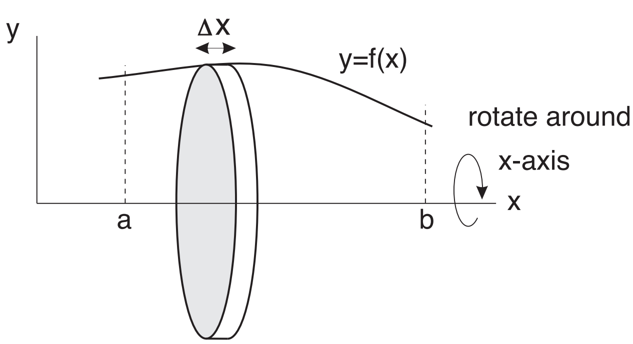
Planar Area

In the limit as $\Delta x \rightarrow dx$ the total number of panels $\rightarrow \infty$

$$A = \int_a^b y \cdot dx = \int_a^b f(x)dx$$

Volume of Solid of Revolution

a) **Disk Method** : rotate $y = f(x)$ about the $x-$ axis to form a solid.



The disk has a volume of $\mathcal{V} = \pi y^2 \Delta x$.

The total volume between a and b can be determined as:

$$\mathcal{V} = \int_a^b \pi y^2 dx$$

Note: The value of $y = f(x)$ is substituted into the formulation for area and the resulting equation is integrated between a and b.

b) **Shell Method:** Find a ring defined with ring area: $2\pi y \cdot \Delta y$.

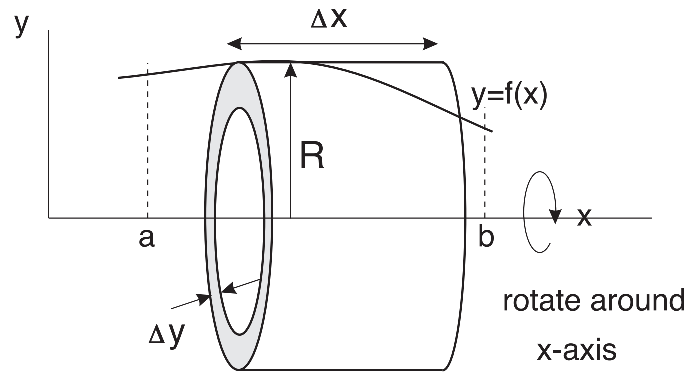
The volume of the ring is given by

$$\Delta \mathcal{V} = (2\pi y \cdot \Delta y)\Delta x$$

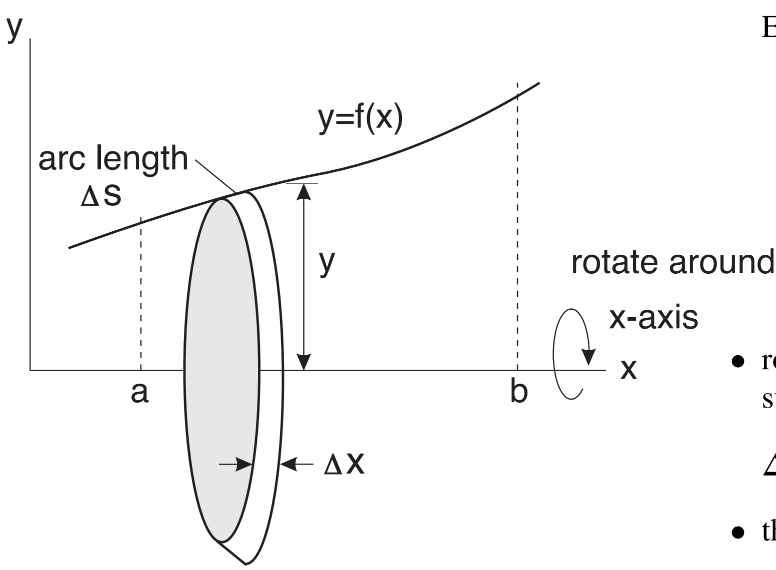
The volume of the solid is determined by solving the integral

$$\mathcal{V} = \int_0^R 2\pi \, x \, y \, dy$$

Either method can be used, which ever is most convenient.



Surface Area of Solid of Revolution



- the arc length can be defined using Eq. 7.15:

$$\Delta s = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\Delta x$$
$$= \frac{\sqrt{dx^2 + dy^2}}{dx}\Delta x$$

- rotate about the $x-$ axis, where the surface area is defined as

$$\Delta A_{surf} = 2\pi y \Delta s$$

- the total surface area is given as

$$A_{surface} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

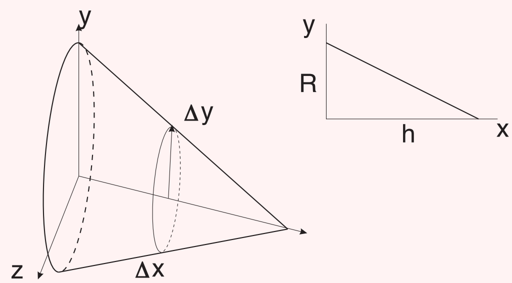
Example: 3.1

Find the area in the positive quadrant bounded by

$$y = \frac{1}{4}x \qquad \text{and} \qquad y = x^3$$

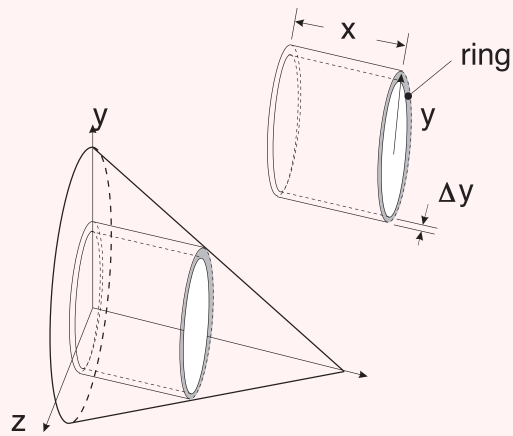
Example: 3.2

Find the volume of a cone with base radius $\boldsymbol{R}$ and height $\boldsymbol{h}$, rotated about the $\boldsymbol{x}$ axis using the disk method.



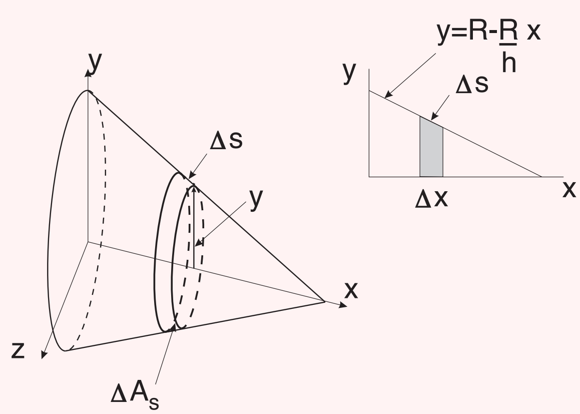
Example: 3.3

Find the volume of a cone with base radius $\boldsymbol{R}$ and height $\boldsymbol{h}$, rotated about the $\boldsymbol{x}$ axis using the shell method.

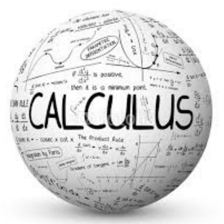


Example: 3.4

Find the surface area of a cone with base radius $\boldsymbol{R}$ and height $\boldsymbol{h}$, rotated about the $\boldsymbol{x}$ axis.



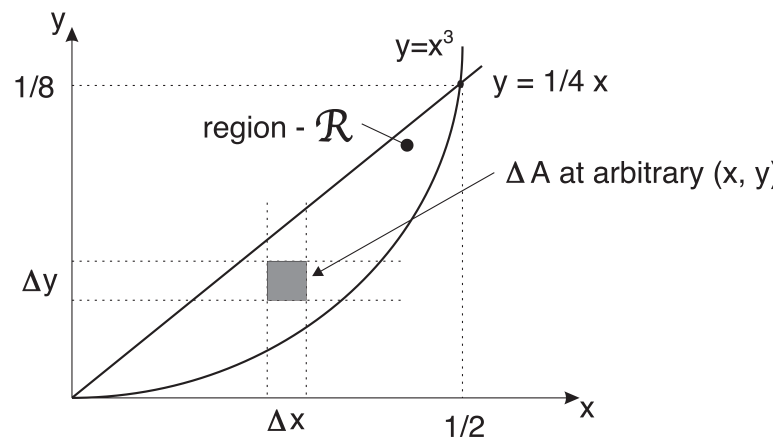
Double Integrals



Reading
Trim 13.1 ⟶ *Double Integrals and Double Iterated Integrals*
13.2 ⟶ *Eval. of Double Integrals by Double Iterated Integrals*
13.7 ⟶ *Double Iterated Integrals in Polar Coordinates*
Assignment
web page ⟶ *assignment #7*

Cartesian Coordinates

Find the area in the +'ve quadrant bounded by $y = \frac{1}{4}x$ and $y = x^3$.



The basic area element in 2D is

$$\Delta A = \Delta x \cdot \Delta y$$

We can build this area up into a strip by summing over Δy, keeping x fixed.

$$\Delta A_{strip} = \left(\sum_{y=x^3}^{1/4x} \Delta y \right)_{fixed\ x} \Delta x$$

Sum up all Δx strips to get the total area

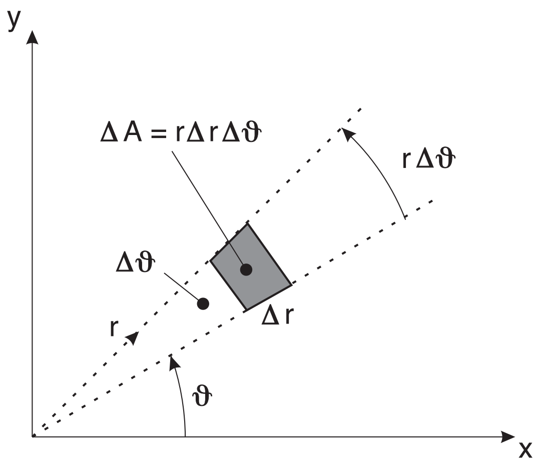
$$A = \sum_{x=0}^{1/2} \left[\sum_{y=x^3}^{1/4x} \Delta y \right] \Delta x$$

In the limit as $\Delta x \to dx$ and $\Delta y \to dy$ we get a double integral as follows

$$A = \int_{x=0}^{1/2} \left(\int_{y=x^3}^{1/4x} dy \right) dx$$

Polar Coordinates

In Cartesian coordinates our area element was $\Delta A = \Delta x \Delta y$, which in differential form gave us



$$A = \int\int_{\mathcal{R}} dx dy$$

We can change the principal coordinates into polar coordinates by transforming x and y into r and θ.

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

The Polar coordinate area element becomes

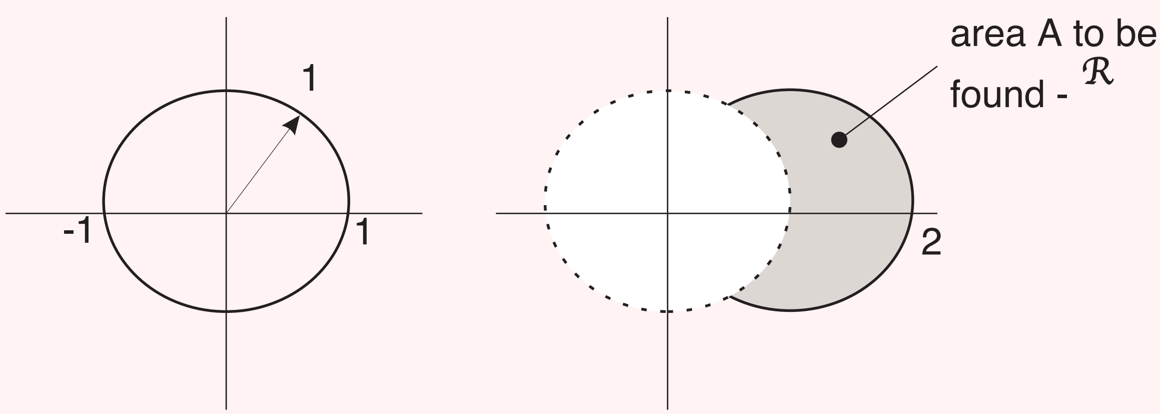
$$\Delta A = r \Delta r \Delta \theta$$

when integrated becomes

$$A = \int\int_{\mathcal{R}} r dr d\theta$$

Example: 3.6

Find the area in the +'ve quadrant bounded by 2 circles



$$\begin{aligned} x^2 + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

Surface Areas from Double Integrals

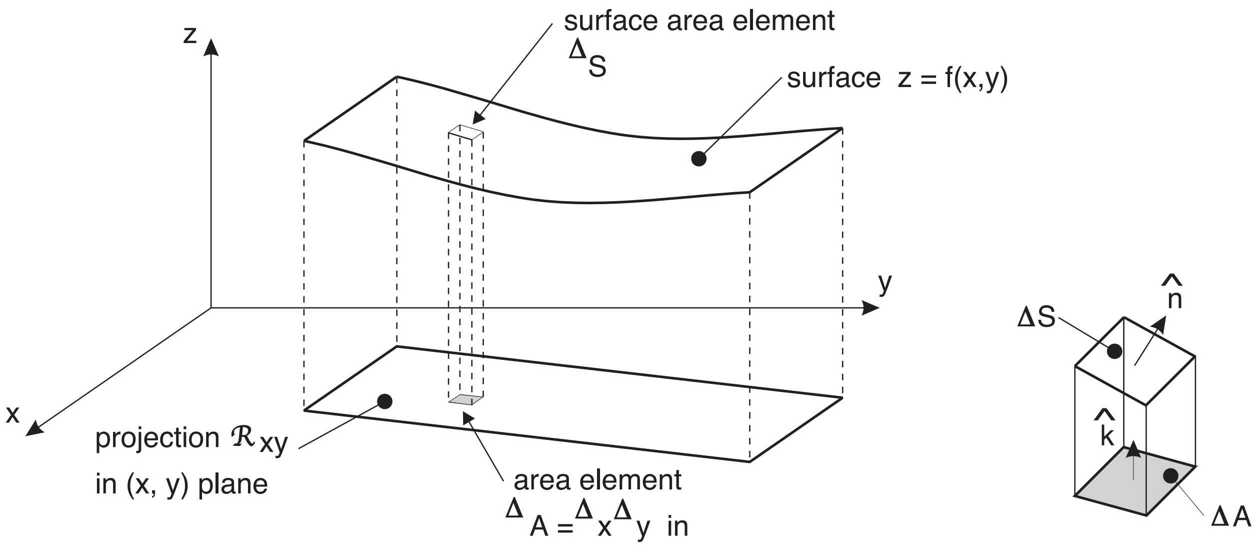
Reading

Trim 13.3 ⟶ *Areas and Volumes of Solids of Revolution*

13.6 ⟶ *Surface Area*

Assignment

web page ⟶ *assignment #7*



How is $\boldsymbol{\Delta S}$ related to $\boldsymbol{\Delta A}$? Imagine shining a light vertically down through $\boldsymbol{\Delta S}$ to get $\boldsymbol{\Delta A}$.

1. the surface is defined as $\boldsymbol{z = f(x, y)}$

2. redefine as $\boldsymbol{F = z - f(x, y)}$ where the surface is given as $\boldsymbol{F = 0}$
   - $\boldsymbol{F > 0}$ and $\boldsymbol{F < 0}$ will the regions above and below the surface, respectively

3. the gradient of the function $\boldsymbol{F}$ is given as

$$\boldsymbol{\nabla F = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)}$$

   $\boldsymbol{\nabla F}$ is the perpendicular to the surface and the perpendicular to the tangent planes

$$\boldsymbol{\vec{n} = \nabla F}$$

4. get the unit normal vector as follows

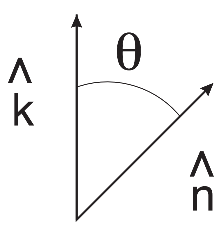
$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{-\hat{i}\dfrac{\partial f}{\partial x} - \hat{j}\dfrac{\partial f}{\partial y} + \hat{k}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + 1}}$$

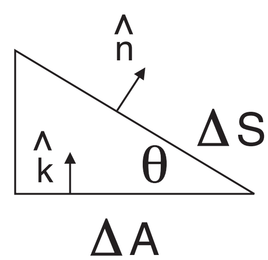
5. find the component of the ΔS surface projected onto $\hat{k}$

   from Trim 12.5 we know that

$$\Delta A = \cos\theta \Delta S$$

   Note, when $\theta = 0 \quad \Rightarrow \quad \Delta A = \Delta S$ (this is the surface parallel to the xy plane.





In general,

$$\Delta A = \underbrace{\cos\theta}_{\hat{n}\cdot\hat{k}} \Delta S$$

$$\hat{n} \cdot \hat{k} = |\hat{n}||\hat{k}| \cos\theta = \cos\theta$$

$$\Delta A = \Delta S(\hat{n} \cdot \hat{k}) = \Delta S \frac{1}{|\nabla F|}$$

since $\hat{n} \cdot \hat{k}$ produces a numerator of $\hat{k} \cdot \hat{k} = 1$ and a denominator of $|\nabla F|$

Rearranging the above equation, we can solve for ΔS. In the limit

$$dS = \underbrace{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}_{|\nabla F|} \underbrace{dxdy}_{dA}$$

Given the surface $z = f(x, y)$, the surface area is

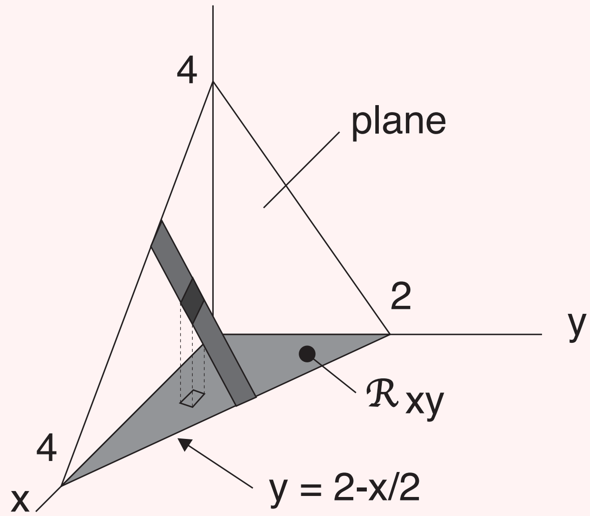
$$\boxed{S = \int\int_{\mathcal{R}_{xy}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy}$$

where $\mathcal{R}_{xy}$ is the projection of the $f(x, y)$ surface down onto the (x, y) plane.

While this is the most common form of the equation, we could also find S by projecting onto another coordinate plane. Sometimes it is more convenient to do it this way. See Trim 14.6 for applicable equations.

Example: 3.7

Find the surface area in the +'ve octant for $z = f(x, y) = 4 - x - 2y$.



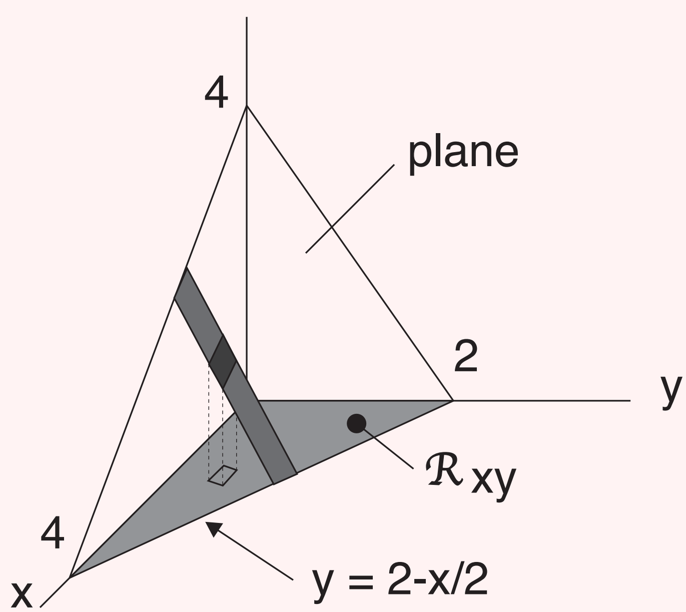
Example: 3.8

Given the sphere, $x^2 + y^2 + z^2 = a^2$, derive the formula for surface area.

Example: 3.9

Find the volume formed in the +'ve octant between the coordinate planes and the surface

$$z = f(x, y) = 4 - x - 2y$$

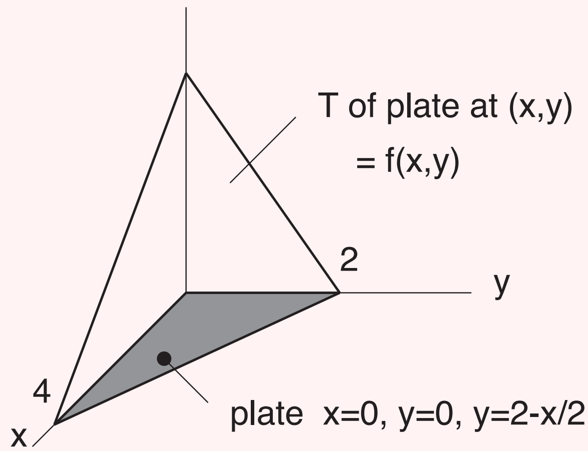


Example: 3.10a

Find the mean value of $y = f(x) = \sin x$ in the domain $x = 0$ to $x = \pi$.

Example: 3.10b

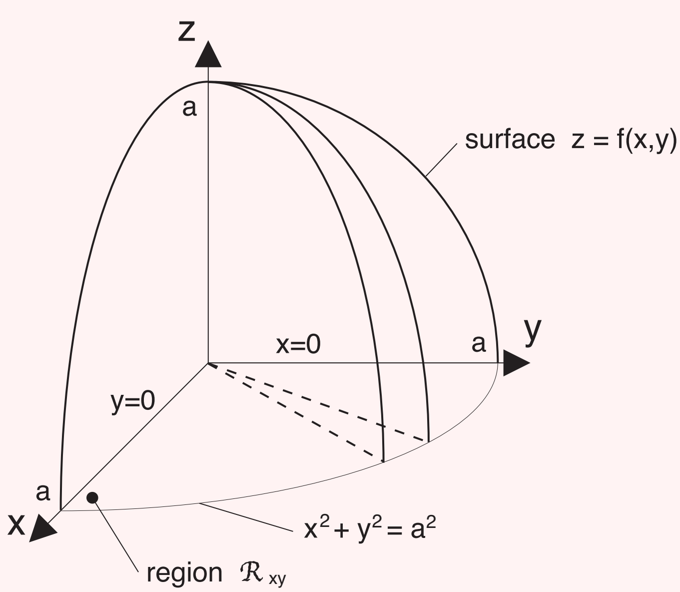
Find the mean value of temperature for $T = f(x, y) = 4 - x - 2y$.



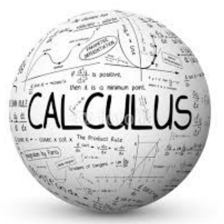
Example: 3.10c

Derive the formula for the volume of revolution. for the following sphere:

$x^2 + y^2 + z^2 = a^2$.



Triple Integrals



Reading

Trim 13.8 ⟶ *Triple Integrals and Triple Iterated Integrals*

13.9 ⟶ *Volumes*

Assignment

web page ⟶ *assignment #8*

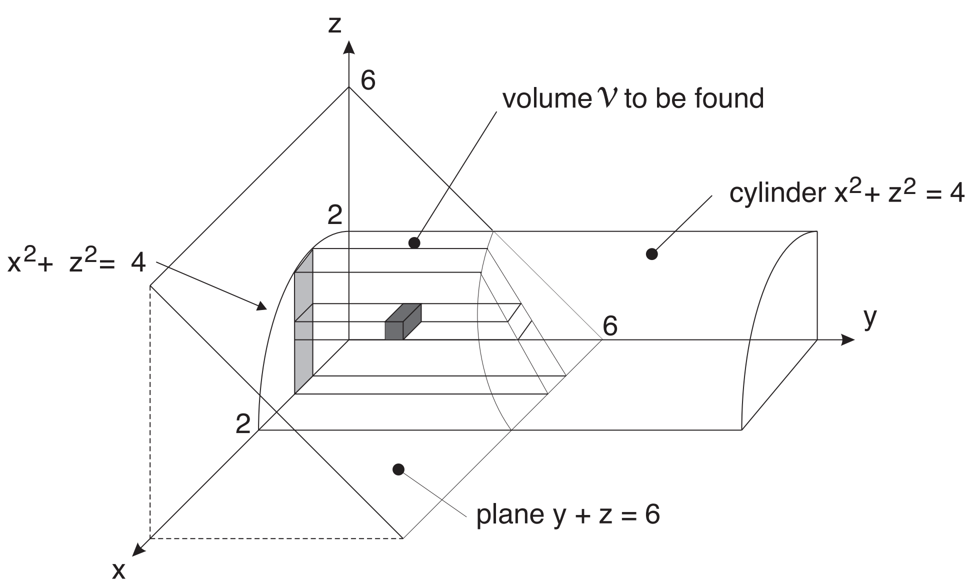
Volume Calculations in Cartesian Coordinates

The triple integral can be identified as

$$\int\int\int_{\mathcal{V}} \underbrace{dx\,dy\,dz}_{d\mathcal{V}\,-\,volume\ element} \qquad \text{or} \qquad \int\int\int_{\mathcal{V}} f\,dx\,dy\,dz$$

add up the $d\mathcal{V}$ elements in x, y, z directions, i.e. a triple sum.

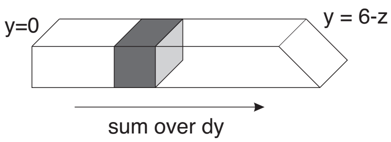
Consider the solid defined by $x^2 + z^2 = 4$ in the positive octant. Find the volume of this solid between the coordinate planes and the plane $y + z = 6$.



Start with a volume element at arbitrary (x, y, z) in space inside $d\mathcal{V}$

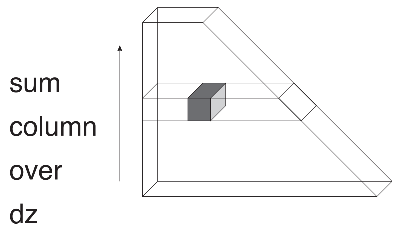
$$d\mathcal{V} = dx\ dy\ dz$$

Build up a column - sum over y keeping x, z constant.



$$\text{column volume} = \left(\int_{y=0}^{6-z} dy\right) dxdz$$

Build up a slice - sum columns over z, keeping y, x fixed.



$$\text{slice volume} = \left[\int_{z=0}^{\sqrt{4-x^2}} \left(\int_{y=0}^{6-z} dy\right) dz\right] dx$$

Finally sum the slices over x

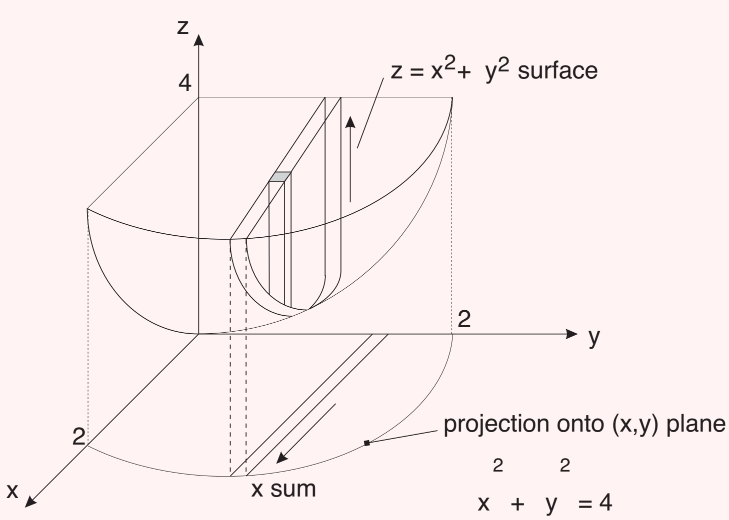
$$\mathcal{V} = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{6-z} dydzdx$$

Evaluation of the integral gives

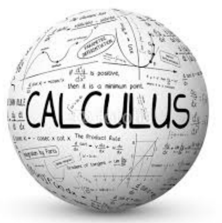
$$\begin{aligned}
\mathcal{V} &= \int_0^2 \int_0^{\sqrt{4-x^2}} (6-z)dzdx = \int_0^2 6\sqrt{4-x^2} - \frac{(\sqrt{4-x^2})^2}{2} dx \\
&= 6\int_0^2 \sqrt{4-x^2}dx - \frac{1}{2}\int_0^2 (4-x^2)dx = 6\pi - \frac{8}{3} \approx 16.18 \qquad \text{use tables if necessary}
\end{aligned}$$

Example: 3.11

Find the volume of the paraboloid, $z = x^2 + y^2$ for $0 \leq z \leq 4$. Consider only the +'ve octant, i.e. $1/4$ of the volume.



Volume Calculations in Cylindrical and Spherical Coordinates



Reading

Trim 13.11 $\longrightarrow$ *Triple Iterated Integrals in Cylindrical Coordinates*

13.12 $\longrightarrow$ *Triple Iterated Integrals in Spherical Coordinates*

Assignment

web page $\longrightarrow$ *assignment #8*

Cylindrical Coordinates

point: $\boldsymbol{P(r, \theta, z)}$ i.e. polar in $\boldsymbol{x, y}$ plane plus $\boldsymbol{z}$

volume element: $\boldsymbol{d\mathcal{V} = r\ dr\ d\theta\ dz}$

based on links to Cartesian coordinates

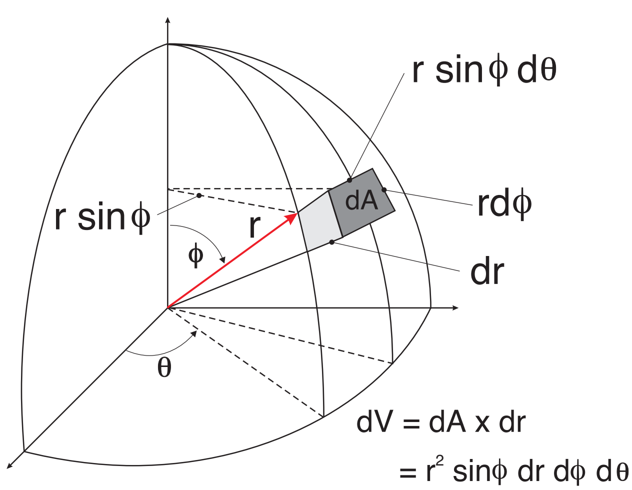
$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}(y/x) \\ z &= z \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

where $\boldsymbol{0 \leq r, z \leq \infty}$ and $\boldsymbol{0 \leq \theta \leq 2\pi}$

Typically we build up column, wedge slice and then the total volume, given as $\int\int\int \boldsymbol{r dr d\theta dz}$

The math operations are easier when we have axi-symmetric systems, i.e. cylinders and cones

Spherical Coordinates



point: $\boldsymbol{P(r, \theta, \phi)}$

volume element: $\boldsymbol{d\mathcal{V} = \underbrace{(r\sin\phi d\phi)}_{height}\underbrace{rdrd\theta}_{area}}$

based on links to Cartesian coordinates

$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \\
\theta &= \tan^{-1}(y/x) \\
\phi &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
x &= r\sin\phi\cos\theta \\
y &= r\sin\phi\sin\theta \\
z &= r\cos\phi
\end{aligned}
$$

where $0 \le r \le \infty$; $0 \le \theta \le 2\pi$; $0 \le \phi \le \pi$. Note: for $0 \le \phi \le \pi$ the $sin\phi$ is always +'ve for $d\mathcal{V}$ +'ve.

The solution procedure involves building up columns, slices as before to obtain the total volume, given as

$$\int\int\int r^2 \ \sin\ \phi\, dr\, d\theta\, d\phi$$

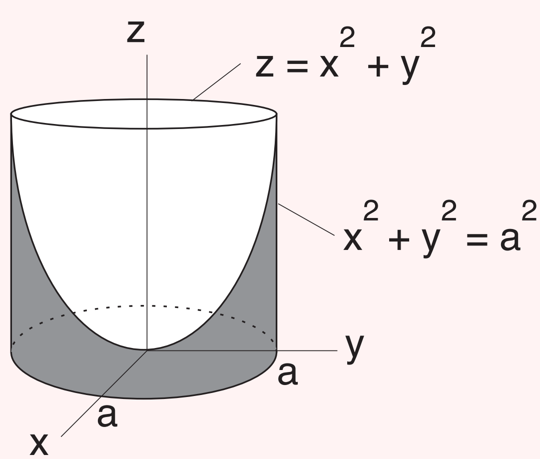
Example: 3.12

Find the volume bounded by a cylinder,

$$x^2 + y^2 = a^2$$

and a paraboloid,

$$z = x^2 + y^2$$



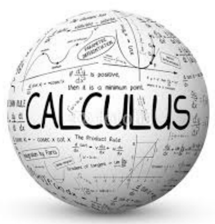
Spherical Coordinate Example

Example: 3.13

Derive a formula for the volume of a sphere with radius, a

$$x^2 + y^2 + z^2 = a^2$$

Moments of Area/ Mass / Volume



Reading

Trim 13.5 ⟶ *Centres of Mass and Moments of Inertia*

13.10 ⟶ *Centres of Mass and Moments of Inertia*

Assignment

web page ⟶ *assignment #9*

Centroids, Centers of Mass etc.

2-D case: thin plate of constant thickness

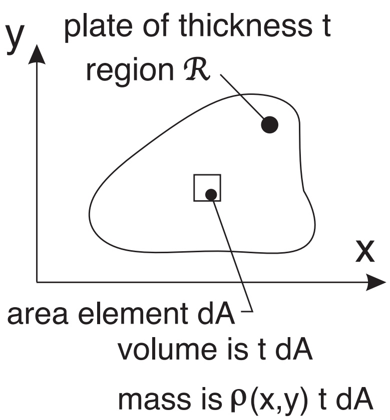
Sometimes, single integrals work, as in a 2-D case, where the thickness is given as t and is constant or a function of position as $t(x, y)$. The material density is given as ρ (kg/m^3), again constant or a function of position as $\rho(x, y)$. We sometimes use the mass per unit area of the plate, $\rho^* = \rho \cdot t$ (kg/m^2).

	area	mass
basic element	$dA = dx\, dy$	$dM = \rho\, t\, dx\, dy$ or $\rho^*\, dx\, dy$
total area	$A = \int\int_{\mathcal{R}} dx\, dy$	$M = \int\int_{\mathcal{R}} dM = \int\int_{\mathcal{R}} \rho\, t\, dx\, dy$
	first moment of area	**first moment of mass**
	(*weight by distance from axis*)	
about $y-$ axis	$x\, dA = x\, dx\, dy$	$x\, dM = x\rho\, t\, dx\, dy$
total	$F_y = \int\int_{\mathcal{R}} x\, dA$	$\int\int_{\mathcal{R}} x\, dM$
about $x-$ axis	$F_x = \int\int_{\mathcal{R}} y\, dA$	$\int\int_{\mathcal{R}} y\, dM$
	centroid coordinates	**center of mass coordinates**
	$\overline{x} = \dfrac{\int\int_{\mathcal{R}} x\, dA}{A}$	$\overline{x_c} = \dfrac{\int\int_{\mathcal{R}} x\, dM}{M}$
	$\overline{y} = \dfrac{\int\int_{\mathcal{R}} y\, dA}{A}$	$\overline{y_c} = \dfrac{\int\int_{\mathcal{R}} y dM}{M}$
second moments		
	$\int\int_{\mathcal{R}} x^2\, dA$	$\int\int_{\mathcal{R}} x^2\, dM$
	$\int\int_{\mathcal{R}} y^2\, dA$	$\int\int_{\mathcal{R}} y^2\, dM$

3-D case:

We use the same basic ideas but the basic element is now $\mathcal{V} = dxdydz$

2-D Objects



Quantities of interest in applications such as dynamics.
Area: $A = \int\int_{\mathcal{R}} dA \; (Volume = tA)$

Mass: $M = \int\int_{\mathcal{R}} \rho(x,y) t dA$
where $\rho(x,y)$ = **density of material** in (kg/m^3) at point (x,y)

Centroid = "geometrical center" of object

$$\overline{x} = \frac{\int\int_{\mathcal{R}} x dA}{A}$$ **1st moment of area** about $y-$ axis

$$\overline{y} = \frac{\int\int_{\mathcal{R}} y dA}{A}$$ **1st moment of area** about $x-$ axis

---

Center of Mass: useful in dynamics problems

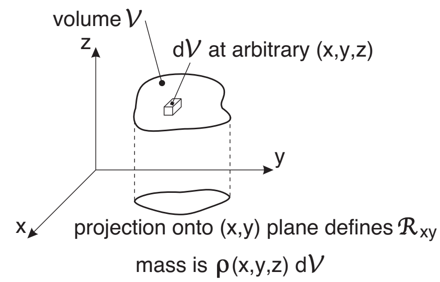
$$\overline{x_c} = \frac{\int\int_{\mathcal{R}} x dm}{M} = \frac{\int\int_{\mathcal{R}} x\rho(x,y) t dA}{M}$$

$$\overline{y_c} = \frac{\int\int_{\mathcal{R}} y dm}{M} = \frac{\int\int_{\mathcal{R}} y\rho(x,y) t dA}{M}$$

Note: that if the object density is uniform, then the centroid and center of mass are the same.

2nd Moments of Area and Mass:
⟶ Moments of Inertia
2nd moment of area about: $y - axis$
$I_y = \int\int_{\mathcal{R}} x^2 dA$
2nd moment of mass about: $y - axis$
$I_y = \int\int_{\mathcal{R}} x^2 \rho(x,y) t dA$
(similar formulas for I_x about the $x - axis$)

3-D Objects



Quantities of interest in applications such as dynamics.
Volume: $\mathcal{V} = \int\int\int_{\mathcal{V}} d\mathcal{V}$

Mass: $M = \int\int\int_{\mathcal{V}} \rho(x,y,z) d\mathcal{V}$
where $\rho(x,y,z)$ = **density of material** in (kg/m^3) at point (x,y,z)

Centroid = "geometrical center" of object

$$\overline{x} = \frac{\int\int\int_{\mathcal{V}} x d\mathcal{V}}{\mathcal{V}}$$ **1st moment of volume** about $y - z$ plane

$$\overline{y} = \frac{\int\int\int_{\mathcal{V}} y d\mathcal{V}}{\mathcal{V}}$$ **1st moment of volume** about $x - z$ plane

$$\overline{z} = \frac{\int\int\int_{\mathcal{V}} z d\mathcal{V}}{\mathcal{V}}$$ **1st moment of volume** about $x - y$ plane

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Center of Mass: useful in dynamics problems

$$\overline{x_c} = \frac{\int\int\int_{\mathcal{V}} x\rho(x,y,z) d\mathcal{V}}{M}$$

similar formulas for $\overline{y}_c$ and $\overline{z}_c$

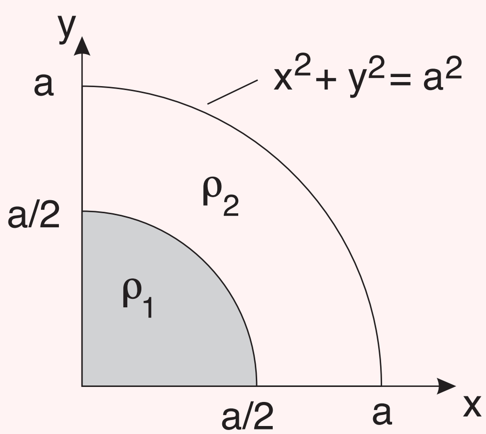
2nd Moments of Area and Mass:
⟶ Polar Moments of Inertia
volume moment about: $y - axis$
$J_y = \int\int\int_{\mathcal{V}} (x^2 + z^2) d\mathcal{V}$
mass moment about: $y - axis$
$J_y = \int\int\int_{\mathcal{V}} (x^2 + z^2)\rho(x,y,z) d\mathcal{V}$
(similar formulas for J_x about the $x - axis$)

and

(similar formulas for J_z about the $z - axis$)

Example: 3.14

Find the centroid, center of mass and the 1st moment of mass for a quarter circle of radius a with an inner circle of radius $a/2$ made of lead with a density of $\rho_1 = 11,000\ kg/m^3$ and an outer circle of radius a made aluminum with a density of $\rho_2 = 2,500\ kg/m^3$. The thickness is uniform throughout at $t = 10\ mm$.



$$\rho_1^* = 11\ g/cm^2 = 110\ kg/m^2$$
$$\rho_2^* = 2.5\ g/cm^2 = 25\ kg/m^2$$

Example: 3.15

Find the area of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$

Example: 3.16

Find the moment of inertia about the $y-$ axis of the area enclosed by the cardioid $r = a(1 - \cos\theta)$

Example: 3.17

Find the center of gravity of a homogeneous solid hemisphere of radius a