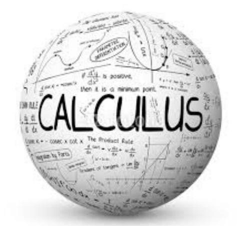
Vectors and Three Dimensional Analytic Geometry

Scalar and Vector Arithmetic



Reading

Trim 11.1 $\longrightarrow$ *Rectangular Coordinates in Space*

11.4 $\longrightarrow$ *Scalar and Vector Products*

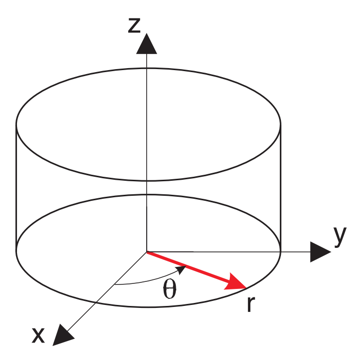
Assignment

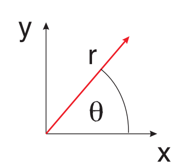
web page $\longrightarrow$ *assignment #1*

Space Coordinates

1. **Cartesian Coordinates:** a system of mutually orthogonal coordinate axes in (x, y, z)
2. **Cylindrical Coordinates:**

   based on the cylindrical coordinate axes (r, θ, z). This is essentially the polar coordinates (r, θ) used instead of (x, y) coupled with the z coordinate.

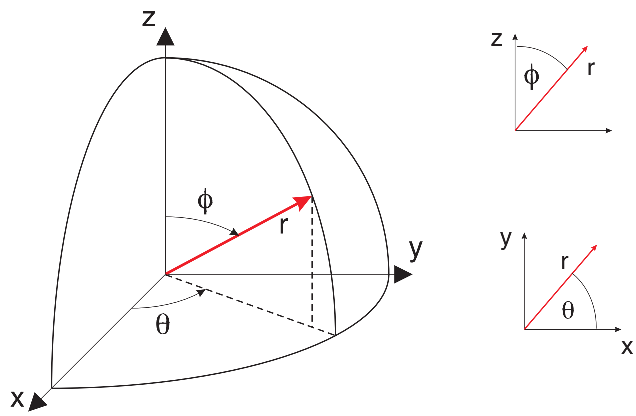




Cylindrical and Cartesian coordinates are related as follows:

Cylindrical to Cartesian	Cartesian to Cylindrical
$x = r \cos \theta$	$r = \sqrt{x^2 + y^2}$
$y = r \sin \theta$	$\theta = \tan^{-1} y/x$
$z = z$	$z = z$

3. **Spherical Coordinates:** based on the spherical coordinate system (r, θ, ϕ), where r is the distance from the origin to the surface of the sphere, ϕ is the angle from the z axis to the radial arm and θ is the angle of rotation about the z axis.

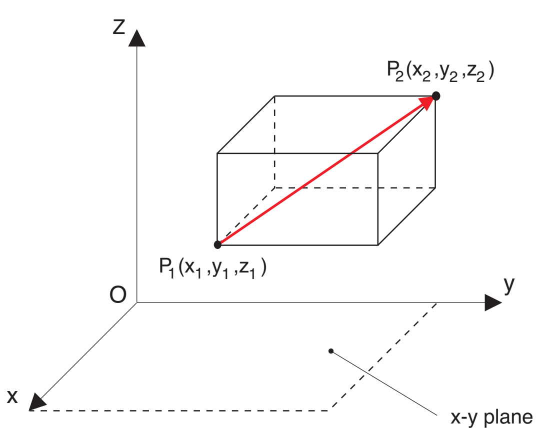


The following relations hold between spherical and Cartesian coordinates.

Spherical to Cartesian	Cartesian to Spherical
$x = r \sin\phi \cos\theta$	$r = \sqrt{x^2+y^2+z^2}$
$y = r \sin\phi \sin\theta$	$\theta = \tan^{-1} y/x$
$z = r \cos\phi$	$\phi = \cos^{-1} z/\sqrt{x^2+y^2+z^2}$

Vectors

Components of Vectors



Vector Notation

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$= \hat{i}(x_2-x_1) + \hat{j}(y_2-y_1) + \hat{k}(z_2-z_1)$$

triple notation

$$= (x_2-x_1,\ y_2-y_1,\ z_2-z_1)$$

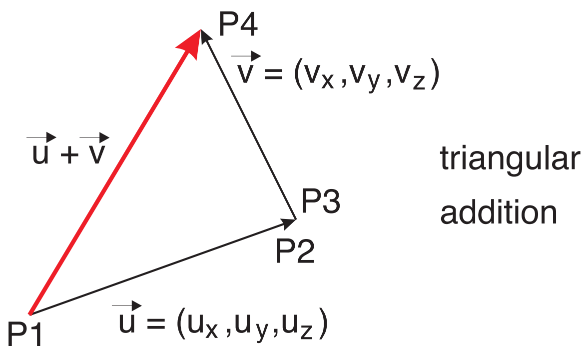
general notation

$$= (v_x,\ v_y,\ v_z)$$

Addition and Subtraction of Vectors

Two vectors $\boldsymbol{u} = (\boldsymbol{u_x}, \boldsymbol{u_y}, \boldsymbol{u_z})$ and $\boldsymbol{v} = (\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z})$ can be added by:

- attaching $\boldsymbol{v} = (\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z})$ to the terminal point of $\boldsymbol{u} = (\boldsymbol{u_x}, \boldsymbol{u_y}, \boldsymbol{u_z})$
- $\boldsymbol{v}$ can be arbitrarily positioned since vector location in space does not matter

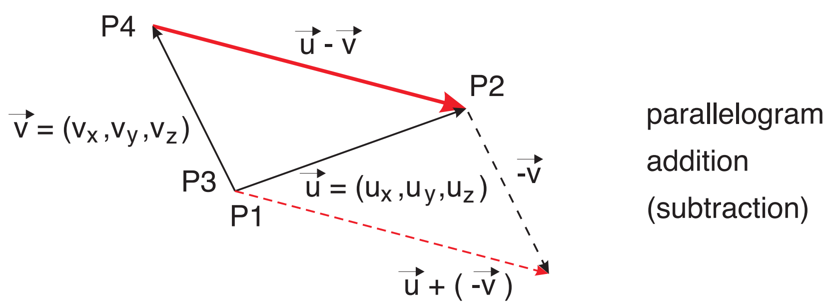


$$\begin{aligned} \boldsymbol{u} &= \overrightarrow{\boldsymbol{P_1P_2}} \\ \boldsymbol{v} &= \overrightarrow{\boldsymbol{P_3P_4}} \\ \boldsymbol{u} + \boldsymbol{v} &= \overrightarrow{\boldsymbol{P_1P_2}} + \overrightarrow{\boldsymbol{P_3P_4}} = \overrightarrow{\boldsymbol{P_1P_4}} \end{aligned}$$

Algebraically, vectors are added component by component, such that

$$\boxed{\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{u_x} + \boldsymbol{v_x}, \boldsymbol{u_y} + \boldsymbol{v_y}, \boldsymbol{u_z} + \boldsymbol{v_z})}$$

Two vectors $\boldsymbol{u} = (\boldsymbol{u_x}, \boldsymbol{u_y}, \boldsymbol{u_z})$ and $\boldsymbol{v} = (\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z})$ are subtracted by attaching $\boldsymbol{u}$ and $\boldsymbol{v}$ to the same starting point. The difference between $\boldsymbol{u}$ and $\boldsymbol{v}$ is given as the vector from the tip of $\boldsymbol{v}$ to the tip of $\boldsymbol{u}$. (or by adding one vector to the negative of the second vector as shown below)



$$\boxed{\boldsymbol{u} - \boldsymbol{v} = (\boldsymbol{u_x} - \boldsymbol{v_x}, \boldsymbol{u_y} - \boldsymbol{v_y}, \boldsymbol{u_z} - \boldsymbol{v_z})}$$

Scalar Multiplication

The multiplication of a vector, $\boldsymbol{v} = (\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z})$ by a positive scalar value, $\boldsymbol{\lambda}$ is obtained by multiplying each component of the vector by the scalar value, such that

$$\boxed{\boldsymbol{\lambda v} = \boldsymbol{\lambda}(\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{v_z}) = (\boldsymbol{\lambda} \cdot \boldsymbol{v_x}, \boldsymbol{\lambda} \cdot \boldsymbol{v_y}, \boldsymbol{\lambda} \cdot \boldsymbol{v_z})}$$

The Scalar Product

The scalar (also referred to as the dot product or the inner product) of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\,\cos\boldsymbol{\theta}$$

where $\boldsymbol{\theta}$ is an angle between $0 \leq \boldsymbol{\theta} \leq \boldsymbol{\pi}$ defined by the vector pair when the initial points coincide.

Note: We can also find the angle between two vectors as: $\cos\boldsymbol{\theta} = \dfrac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}$

Useful Properties

	$\boldsymbol{A}(a_x, a_y, a_z) \qquad \boldsymbol{B}(b_x, b_y, b_z)$
Scalar Product	$\boldsymbol{A} \cdot \boldsymbol{B} = a_x b_x + a_y b_y + a_z b_z$
Commutative law	$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$
Distributive law	$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$
	$(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C}$
	$(\boldsymbol{A} + \boldsymbol{B}) \cdot (\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{A} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{D}$
Orthogonal vectors	$\boldsymbol{A} \cdot \boldsymbol{B} = 0$ if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular (since $\cos\ 90^o = 0$)
Coincident vectors	$\boldsymbol{B} = \boldsymbol{A}$, then $\boldsymbol{\theta} = 0$ and $\cos\boldsymbol{\theta} = 1 \;\rightarrow\; \boldsymbol{A} \cdot \boldsymbol{A} = |\boldsymbol{A}|^2$

Note:

$\boldsymbol{\theta} = 0$ vectors are parallel

$\boldsymbol{\theta} = \boldsymbol{\pi}/2\ (\cos\boldsymbol{\theta} = 0)$ vectors are perpendicular

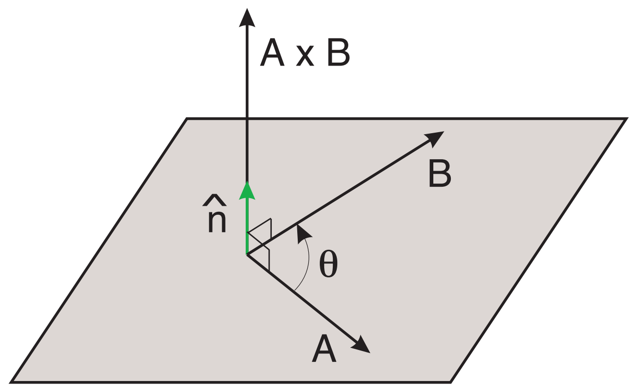
The Vector Product

Also referred to as the cross product or the outer product.

Two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ sharing the same origin and separated by a angle $\boldsymbol{\theta}$ form a plane. If we let $\vec{\boldsymbol{n}}$ be a unit vector perpendicular to this plane, pointing in a direction dictated by the right hand rule, the vector or cross product can be defined as

$$\boxed{\boldsymbol{A} \times \boldsymbol{B} = \hat{\boldsymbol{n}}|\boldsymbol{A}|\,|\boldsymbol{B}|\,\sin\boldsymbol{\theta}}$$

$\hookrightarrow$ since it is a vector we must give it direction



Useful Properties

$$\boldsymbol{A}(a_x, a_y, a_z) \qquad \boldsymbol{B}(b_z, b_y, b_z)$$

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$$

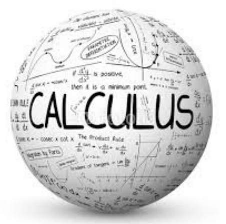
$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$$

$$(\boldsymbol{B} + \boldsymbol{C}) \times \boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{A} + \boldsymbol{C} \times \boldsymbol{A}$$

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\hat{i}(a_y b_z - a_z b_y) - \hat{j}(a_x b_z - a_z b_x) + \hat{k}(a_x b_y - a_y b_x)$$

Vectors and Points, Curves, Lines in 3-D

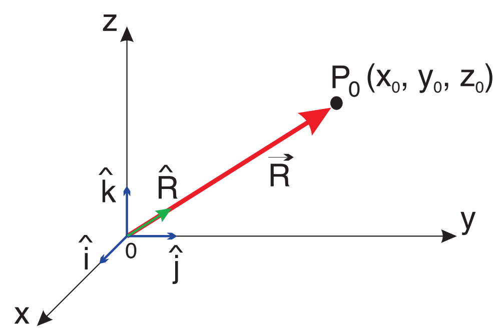


Reading
Trim 11.2 $\longrightarrow$ *Curves and Surfaces*
11.3 $\longrightarrow$ *Vectors*
11.5 $\longrightarrow$ *Planes and Lines*
Assignment
web page $\longrightarrow$ *assignment #1*

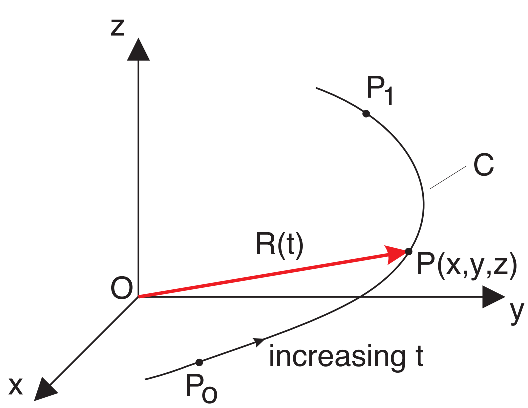
Point in 3D

The position vector for this point is given by the directed line segment $\overrightarrow{OP_0}$.

$$\begin{aligned}
\vec{R} &= \hat{i}x_0 + \hat{j}y_0 + \hat{k}z_0 \\
&= \left|\vec{R}\right| \hat{R} = \sqrt{x_0^2 + y_0^2 + z_0^2}\ \hat{R} \\
&= \text{magnitude of } \vec{R} \ \times \ \text{unit} \\
&\qquad \text{vector in } \vec{R} \text{ direction}
\end{aligned}$$



Curve in 3D



Imagine that point P moves through 3D space in time. These equations are denoted as the parametric equations of motion for the particle and we refer to C as the trajectory of the particle in 3D space.

The position vector is now a function of t as well.

$$\vec{R}(t) = \hat{i}\ x(t) + \hat{j}\ y(t) + \hat{k}\ z(t)$$

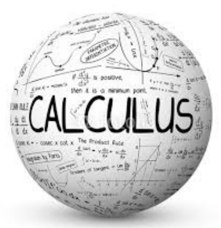
Example 1.1

Suppose P moves from $(-1, 0, 1)$ to $(2, 2, -1)$ in 4 seconds in a straight line and the x coordinate increases linearly with time. i.e. $x(t) = at + b$

a) find the position vector vs. time, for points on the line

b) find the equation of the line in 3D space

Vectors and Planes in 3-D



Reading

Trim 11.5 $\longrightarrow$ *Planes and Lines*

11.6 $\longrightarrow$ *Geometric Applications of Scalar and Vector Products*

Assignment

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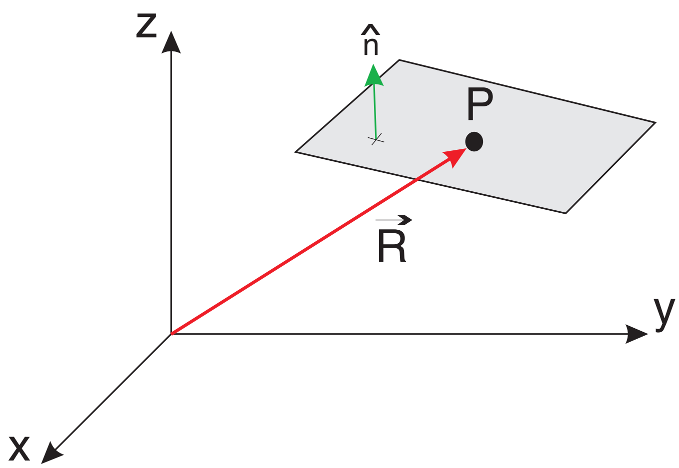
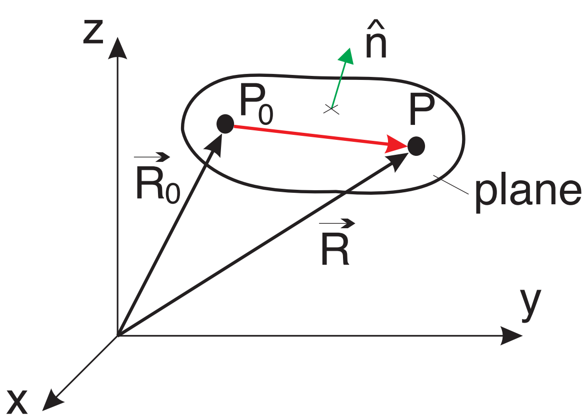
Suppose point $\boldsymbol{P}$, with position vector $\vec{\boldsymbol{R}}$, lies in a plane in 3D space. One way to define the plane is through a unit normal vector, $\hat{\boldsymbol{n}}$.





$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{\imath}} n_x + \hat{\boldsymbol{\jmath}} n_y + \hat{\boldsymbol{k}} n_z$$

If we assume there is a fixed point, $\boldsymbol{P}_0(x_0, y_o, z_0)$, lying **in the plane** with position vector

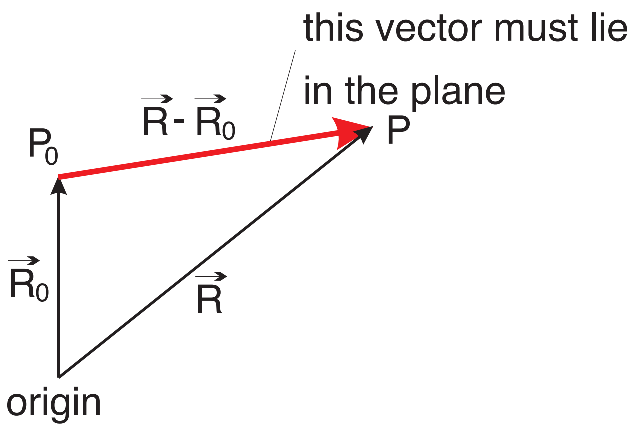
$$\vec{\boldsymbol{R}}_0 = \hat{\boldsymbol{\imath}} x_0 + \hat{\boldsymbol{\jmath}} y_0 + \hat{\boldsymbol{k}} z_0$$

We can find the equation for the arbitrary point $\boldsymbol{P}(x, y, z)$ **in the plane**, by first defining the position vector to $\boldsymbol{P}$ as

$$\vec{\boldsymbol{R}} = \hat{\boldsymbol{\imath}} x + \hat{\boldsymbol{\jmath}} y + \hat{\boldsymbol{k}} z$$

Consider

$$\vec{R} - \vec{R}_0 = \hat{i}(x - x_0) + \hat{j}(y - y_0) + \hat{k}(z - z_0)$$



For the points to be in the plane, $\vec{R} - \vec{R}_0$ must be perpendicular to $\hat{n}$. Therefore

$$(\vec{R} - \vec{R}_0) \cdot \hat{n} = 0$$

The dot product gives

$$\left[\underbrace{\hat{i}(x - x_0) + \hat{j}(y - y_0) + \hat{k}(z - z_0)}_{\vec{R} - \vec{R}_0}\right] \cdot \left[\underbrace{\hat{i}n_x + \hat{j}n_y + \hat{k}n_z}_{\hat{n}}\right] = 0$$

which leads to

$$\boxed{n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0}$$

This is the general equation for a plane through a point (x_0, y_0, z_0) that is perpendicular to (n_x, n_y, n_z).

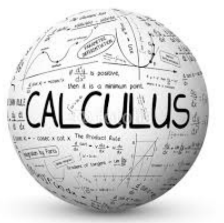
Example 1.2

Given that points $(1, 0, 0)$ $(0, 3, 0)$ $(0, 0, 2)$ lie on a plane in 3D, find

a) the unit normal vector $\hat{n}$ to the plane

b) the equation describing points (x, y, z) lying in the plane

Calculation of Distance in 3-D



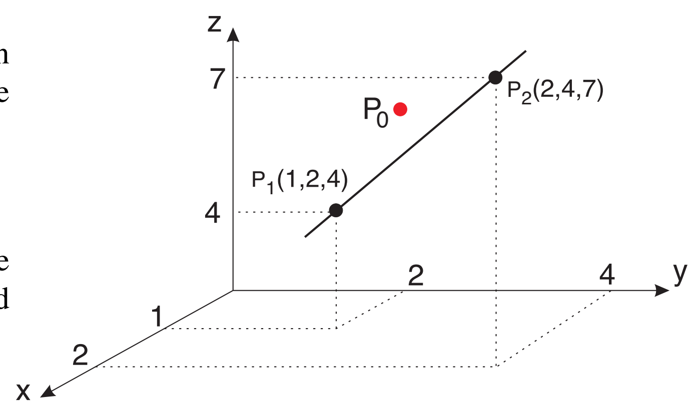
Reading
Trim 11.6 $\longrightarrow$ *Geometric Applications of Scalar and Vector Products*

Assignment
web page $\longrightarrow$ *assignment #2*

i) shortest distance between a point and a line

3 steps are required to find the shortest distance between a point and a line:

1. determine the position vector between an arbitrary point on the line and the point of interest itself
2. find the unit vector along the line
3. the shortest distance is the length of the cross product of the position vector and the unit vector along the line



We will use the case of robot arm moving along a straight line to illustrate the technique. The equation of the straight line is given in symmetric form as

$$x - 1 = \frac{y-2}{2} = \frac{z-4}{3}$$

How close does it pass to the point $P_0(1, 3, 6)$?

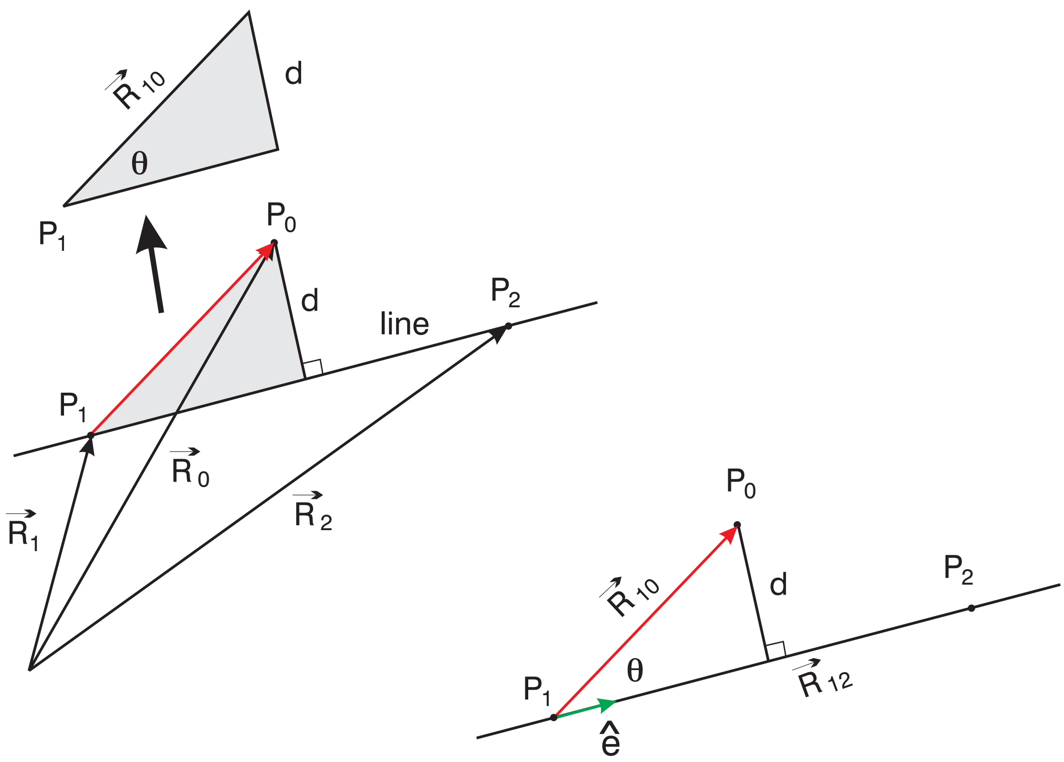
Step 1: Find a position vector between the line and the point

set $x = 1$ to find point P_1 and its position vector

$$P_1 : \quad \vec{R}_1 = \hat{i} + 2\hat{j} + 4\hat{k} \qquad \Rightarrow \qquad P_1(1, 2, 4)$$

find the position vector to our point of interest

$$P_0 : \quad \vec{R}_0 = \hat{i} + 3\hat{j} + 6\hat{k} \qquad \Rightarrow \qquad P_0(1, 3, 6)$$



The position from the line to the point is

$$\vec{R}_{10} = \vec{R}_0 - \vec{R}_1 = (0)\hat{\imath} + \hat{\jmath} + (2)\hat{k}$$

Step 2: Find the unit vector along the line

The position vector to a second point along the line can be determined by letting $x = 2$

$$P_2 : \quad \vec{R}_2 = 2\hat{\imath} + 4\hat{\jmath} + 7\hat{k} \qquad \Rightarrow \qquad P_2(2, 4, 7)$$

By definition, the unit vector along the line, $\hat{e}$, is given as the vector divided by its magnitude

$$\hat{e} = \frac{\vec{R}_2 - \vec{R}_1}{\left|\vec{R}_2 - \vec{R}_1\right|} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|}$$

where

$$\vec{R}_{12} = \vec{R}_2 - \vec{R}_1 = \hat{\imath} + (2)\hat{\jmath} + (3)\hat{k}$$

Step 3: Find the shortest distance between the point and the line

Method 1: Cross Product Approach

$$\boxed{\boldsymbol{d = \frac{|\vec{R}_{10} \times \vec{R}_{12}|}{|\vec{R}_{12}|}}}$$

Method 2: Dot Product Approach

$$\boxed{\boldsymbol{d = \left|\vec{R}_{10}\right| \, \sin\theta}}$$

Using the cross product approach:

$$\boldsymbol{|\vec{R}_{10} \times \vec{R}_{12}| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}}$$

$$\boldsymbol{|\vec{R}_{12}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}}$$

The magnitude of $\boldsymbol{d}$ is

$$\boldsymbol{\text{mag } d \;\; = \;\; \frac{\sqrt{6}}{\sqrt{14}} = 0.655}$$

ii) shortest distance between a point and a plane

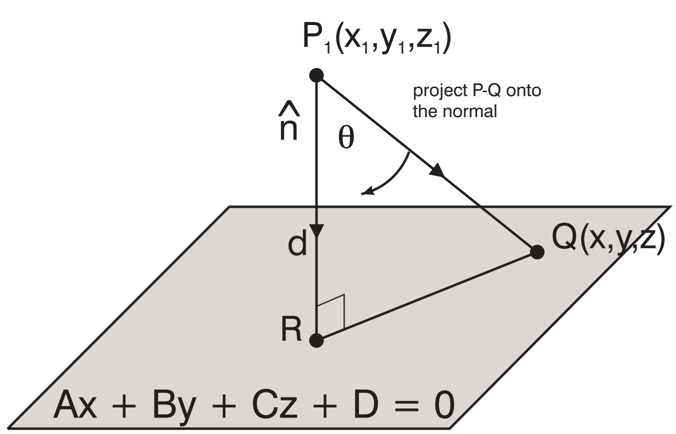
3 steps are required to find the shortest distance between a point and a plane:

1. determine a vector between an arbitrary point in space and a point on the plane
2. find the unit normal from the point to the plane
3. use the dot product of the vector and the unit normal to find the shortest distance

The equation of a plane is given as

$$\boldsymbol{Ax + By + Cz + D = 0}$$

Let $\boldsymbol{P_1(x_1, y_1, z_1)}$ be **any point in space** and $\boldsymbol{Q(x, y, z)}$ be a **point on the plane** specified above as shown below



Step 1: Determine the vector between the point in space and the point on the plane

$$\vec{PQ} = (x - x_1), (y - y_1), (z - z_1)$$

Step 2: Determine the unit normal vector between the point in space and the plane

$$\text{normal to plane} \;=\; (A, B, C)$$

$$\hat{PR} \;=\; \frac{\vec{PR}}{|\vec{PR}|} = \frac{\pm(A, B, C)}{\sqrt{A^2 + B^2 + C^2}}$$

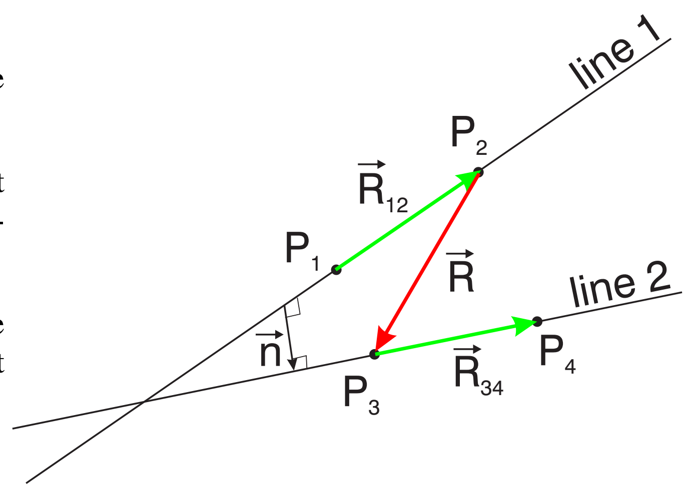
Step 3: Use the dot product approach to find the shortest distance

$$\boxed{d = \frac{|Ax_1 \;+\; By_1 \;+\; Cz_1 \;+\; D|}{\sqrt{A^2 + B^2 + C^2}}}$$

iii) shortest distance between two skewed lines

3 steps are required to find the shortest distance between two non-instersecting lines:

1. find the unit normal vector between the two lines, $\hat{n}$
2. determine a position vector staring at any point on the first line and terminating at any point on the second line
3. the shortest distance is the length of the position vector projected onto the unit normal vector



We will use the case of the following two non-intersecting, skewed lines to illustrate.

line 1 $\quad x = y = z$

line 2 $\quad y = 2 \quad z = 0$

Step 1: Find the unit normal vector (either dot product <u>or</u> cross product approach

<u>Dot Product Approach</u> $\Rightarrow$ $\cos\theta = \dfrac{A \cdot B}{|A|\,|B|} = 0$ (for $\perp$)

We know that the dot product of a line and its normal equals zero.

$$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$$

If $(0, 0, 0)$ and $(1, 1, 1)$ are two points on line 1, then

$$n_x + n_y + n_z = 0 \qquad (1)$$

If $(1, 2, 0)$ and $(0, 2, 0)$ are two points on line 2, then

$$n_x = 0 \qquad (2)$$

Therefore $n_x = 0, n_y = 1$ and $n_z = -1$ is a solution and

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{n_x\,\hat{i} + n_y\,\hat{j} - n_z\,\hat{k}}{\sqrt{n_y^2 + n_z^2}} = \frac{1}{\sqrt{2}}\,(\hat{j} - \hat{k})$$

<u>Cross Product Approach</u> $\Rightarrow$ $\hat{n} = \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{A \times B}{|A|\,|B| \sin\theta} = \dfrac{A \times B}{|\vec{n}|}$

2 points on line 1 $\quad P_1(0, 0, 0)$ and $P_2(1, 1, 1)$

vector on line 1 $\quad \vec{R}_{21} = \vec{R}_2 - \vec{R}_1 = \hat{i} + \hat{j} + \hat{k}$

2 points on line 2 $\quad (0, 2, 0)$ and $(1, 2, 0)$

vector on line 2 $\quad \vec{R}_{43} = \vec{R}_4 - \vec{R}_3 = +\hat{i}$

$$\vec{n} = \underbrace{(\hat{i} + \hat{j} + \hat{k})}_{vector\ 1} \times \underbrace{\hat{i}}_{vector\ 2} = \hat{i}(0) - \hat{j}(0 - 1) + \hat{k}(0 - 1) = \hat{j} - \hat{k}$$

and we know

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{j} - \hat{k}}{\sqrt{2}}$$

Step 2: Find a position vector between points on each line

The position vector between any 2 points on the 2 lines, say $(0,0,0)$ and $(1,2,0)$ is

$$\vec{R} = \hat{i} + 2\hat{j}$$

Step 3: Find the shortest distance, d

Dot Product

$\vec{n} \cdot$ line 1 $= 0$

$\vec{n} \cdot$ line 2 $= 0 \;\Rightarrow\;$ solve for $n_x,\ n_y,\ n_z$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|}$$

$d = |\text{line 3} \cdot \hat{n}|$

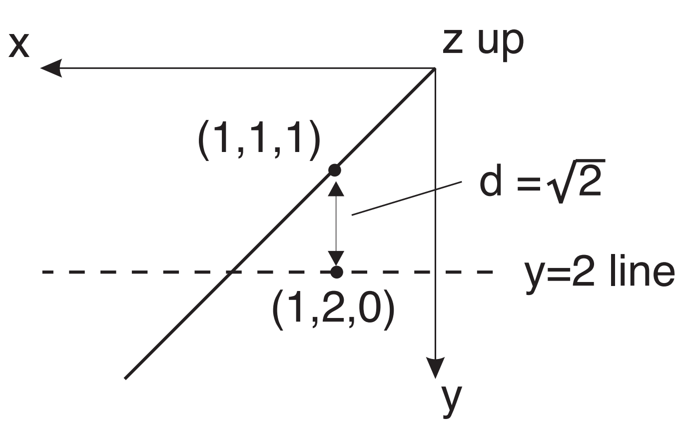
Cross Product

$$\hat{n} = \frac{\text{line 1} \times \text{line 2}}{|\text{line 1} \times \text{line 2}|}$$

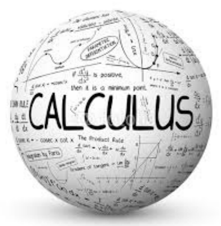
$d = |\text{line 3} \cdot \hat{n}|$

Dot Product Approach

$$d = \left| (\hat{i} + 2\hat{j}) \cdot \frac{\hat{j} - \hat{k}}{\sqrt{2}} \right| = \frac{2}{\sqrt{2}} = \sqrt{2}$$



Differentiation of Vectors



Reading
Trim 11.9 $\longrightarrow$ *Differentiation and Integration of Vectors*

Assignment
web page $\longrightarrow$ *assignment #2*

Think of a dynamics problem of a particle moving along some curve in 3D space where a twist may be possible, i.e. not planar. The particles position is defined by a position vector

$$\vec{R}(t) = \hat{\imath}\, x(t) + \hat{\jmath}\, y(t) + \hat{k}\, z(t) \qquad \text{for } \ a \leq t \leq b$$

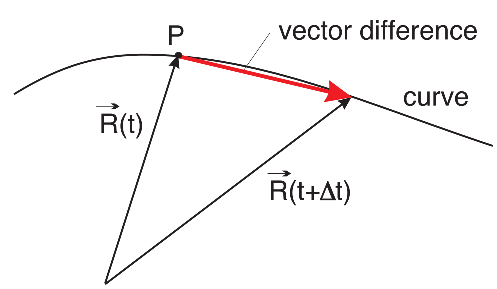
The formal definition of the derivative can be written as

$$\left.\frac{d\vec{R}}{dt}\right|_{at\ P} = \lim_{\Delta t \to 0} \frac{\vec{R}(t+\Delta t) - \vec{R}(t)}{\Delta t}$$

How do we calculate this from the definition?

$$\begin{aligned}
\left.\frac{d\vec{R}}{dt}\right|_{at\ P} &= \lim_{\Delta t \to 0} \frac{\hat{\imath}x(t+\Delta t) + \hat{\jmath}y(t+\Delta t) + \hat{k}z(t+\Delta t) - \hat{\imath}x(t) - \hat{\jmath}y(t) - \hat{k}z(t)}{\Delta t} \\
&= \hat{\imath} \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} + \hat{\jmath} \lim \Big(\Big) + \hat{k} \lim \Big(\Big) \\
&= \hat{\imath} \frac{dx(t)}{dt} + \hat{\jmath} \frac{dy(t)}{dt} + \hat{k} \frac{dz(t)}{dt}
\end{aligned}$$

We can just differentiate each component in the position vector $\vec{R}(t)$. The differentiation process produces another vector. The physical meaning of differentiation is that we have a slope but not just a slope – it is tied back to the particle motion where $\vec{R}(t+\Delta t) - \vec{R}(t)$ is a vector difference that in the limit will give us the tangent vector at point P.



Therefore we must have the rate of change position vector along the curve in the $\hat{\boldsymbol{T}}$ direction. (Let $\hat{\boldsymbol{T}}$ be the unit vector in the tangent direction)

$$\frac{d\vec{R}}{dt} = \left|\frac{d\vec{R}}{dt}\right| \hat{T}$$

Given the particle trajectory for curve $\vec{R}(t)$

$$\left.\frac{d\vec{R}}{dt}\right|_{at\ P} = \hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt} = \left|\vec{V}\right| \hat{T}$$

where the velocity of the particle, $\vec{V}$, is the rate of change of position.

Similar ideas apply for higher derivatives.

2nd derivative of $\vec{R}(t)$

It is easy to see

$$\left.\frac{d^2\vec{R}}{dt^2}\right|_{at\ P} = \hat{i}\frac{d^2x}{dt^2} + \hat{j}\frac{d^2y}{dt^2} + \hat{k}\frac{d^2z}{dt^2}$$

Since we have just shown that

$$\frac{d\vec{R}}{dt} = \left|\frac{d\vec{R}}{dt}\right| \hat{T} = \left|\vec{V}\right| \hat{T}$$

Then the second derivative is

$$\begin{aligned} \frac{d^2\vec{R}}{dt^2} &= \frac{d}{dt}\left(\frac{d\vec{R}}{dt}\right) = \frac{d}{dt}\left(\left|\vec{V}\right| \hat{T}\right) \\ &= \underbrace{\hat{T}\frac{d\left|\vec{V}\right|}{dt}}_{acceleration} + \left|\vec{V}\right| \frac{d\hat{T}}{dt} \end{aligned}$$

The first term, $\hat{T}\frac{d\left|\vec{V}\right|}{dt}$, is the **acceleration component along the curve** in the $\hat{\boldsymbol{T}}$ direction, i.e. the speed change along the curve.

The second term is the **centripetal acceleration** due to a changing trajectory path

$$\left|\vec{V}\right|\left|\frac{d\hat{T}}{dt}\right|\hat{N}$$

where $\hat{N}$ is called the principal normal unit vector.

Therefore for a given curve

$$\frac{d^2\vec{R}}{dt^2}=\hat{i}\frac{d^2x}{dt^2}+\hat{j}\frac{d^2y}{dt^2}+\hat{k}\frac{d^2z}{dt^2}=\hat{T}\frac{d\left|\vec{V}\right|}{dt}+\left|\vec{V}\right|\left|\frac{d\hat{T}}{dt}\right|\hat{N}$$

where

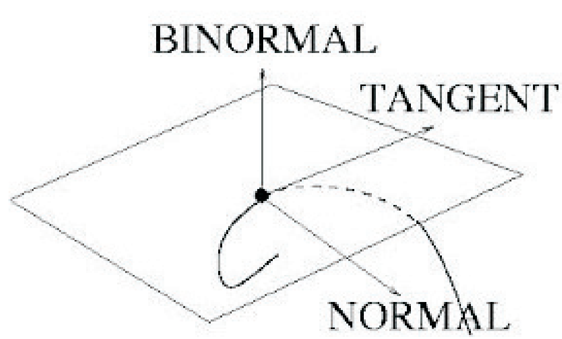
1st term	velocity acceleration at P
2nd term	Cartesian components of acceleration
3rd term	components of acceleration in coordinates relative to the trajectory - speed along the trajectory plus the centripetal acceleration due to trajectory shape

Note: we can turn this around and use to calculate the principal normal to any given curve at $\vec{R}(t)$

$$\hat{N}=\frac{\frac{d\hat{T}}{dt}}{\left|\frac{d\hat{T}}{dt}\right|}$$

where

$$\hat{T}=\frac{\frac{d\hat{R}}{dt}}{\left|\frac{d\hat{R}}{dt}\right|}$$

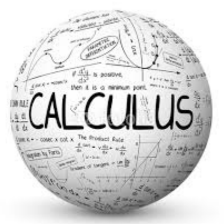


and

$$\hat{B}=\hat{T}\times\hat{N}$$

gives the third local coordinate.

Integration of Vectors



Reading
Trim 11.9 $\longrightarrow$ *Differentiation and Integration of Vectors*

Assignment
web page $\longrightarrow$ *assignment #2*

Some of the ideas learned in first year calculus carry over to vectors in 3-D space.

For a moving particle, the position vector $\vec{R}(t)$ and its derivative $d\vec{R}(t)/dt = \vec{V}(t) = \left|\vec{V}(t)\right| \hat{T}(t)$ can be used.

We see that

$$\int \frac{d\vec{R}(t)}{dt} \cdot dt = \int \vec{V}(t) dt = \vec{R}(t) + \vec{C}$$

where $\vec{C}$ is the vector constant.

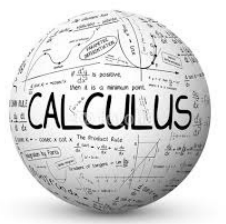
Example 1.3

Given a particle at $t = 0$ and position $(2, 0, 2)$ that moves with a velocity

$$\vec{V}(t) = \hat{\imath}(-2\sin t) + \hat{\jmath}(2\cos t) + \hat{k}(2\sin t - 2\cos t)$$

for $0 \leq t \leq 2\pi$, find the curve that $\vec{R}(t)$ traces in 3D space.

Tangent Vectors and Arc Length of Curves in 3-D



Reading

Trim 11.10 $\longrightarrow$ *Parametric and Vector Representation of Curves*

11.11 $\longrightarrow$ *Tangent Vectors and Lengths of Curves*

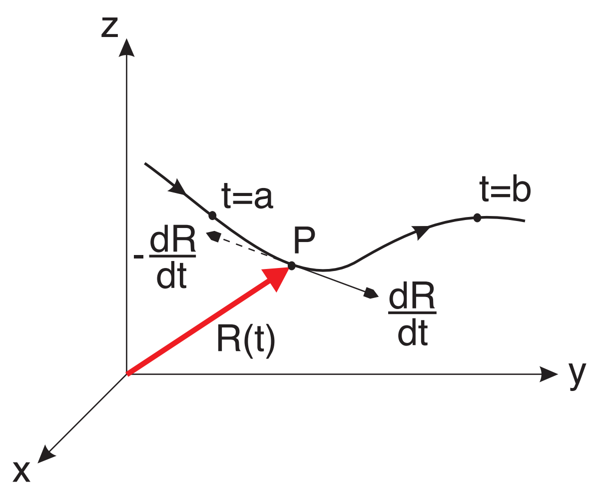
Assignment

webpage $\longrightarrow$ *assignment #2 & #3*

The tangent vector to the curve at point $\boldsymbol{P}$ is given by

$$\vec{T} = \frac{d\vec{R}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

There are two possible tangent directions at each point, however, the tangent vector is always defined in the direction of increasing t along the curve as shown below.



To find the unit tangent vector to a curve, $\hat{T}$ at any point along the curve, the tangent vector is normalized with respect to its length as follows

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{d\vec{R}/dt}{|d\vec{R}/dt|}$$

We recall that for the 2D case that the length is given by

$$\begin{aligned}\Delta L^2 &= \Delta x(t)^2 + \Delta y(t)^2 \\ \Delta L &= \sqrt{\Delta x(t)^2 + \Delta y(t)^2} = \sqrt{\frac{\Delta x(t)^2}{\Delta t^2} + \frac{\Delta y(t)^2}{\Delta t^2}}\,\Delta t\end{aligned}$$

In differential form this becomes

In differential form this becomes

$$dL = \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2}\,dt$$

In general

$$\begin{aligned} L &= \int_{x=x_0}^{x_1} \sqrt{\frac{dx^2 + dy^2}{dx^2}}\, dx \\ &= \int_{x=x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \end{aligned}$$

Summing all the differential lengths, starting at $t = a$

$$L = \int_{t=a} \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2} dt \qquad \text{or}$$

We can extend this to 3D space, where the position vector

$$\vec{R}(t) = \hat{\imath}x(t) + \hat{\jmath}y(t) + \hat{k}z(t)$$

for $a \leq t \leq b$.

If we consider a small section of the curve for a small Δt

$$\Delta S \approx \left|\vec{R}(t + \Delta t) - \vec{R}(t)\right|$$

If we divide through by Δt

$$\frac{\Delta S}{\Delta t} = \left|\frac{\vec{R}(t + \Delta t) - \vec{R}(t)}{\Delta t}\right|$$

In the limit as $\Delta t \to 0$ the equality is exact, therefore

$$\frac{dS}{dt} = \left|\frac{d\vec{R}(t)}{dt}\right|$$

Note the signs, +'ve ΔS in the direction of +'ve Δt.

We have

$$\frac{dS}{dt} = \left|\frac{d\vec{R}}{dt}\right| = \left|\hat{i}\frac{dx(t)}{dt} + \hat{j}\frac{dy(t)}{dt} + \hat{k}\frac{dz(t)}{dt}\right|$$

Therefore

$$\frac{dS}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

or

$$dS = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

where dS is the arc length along the curve.

The total length of the curve is given by

$$L = \int_a^b dS = \int_{t=a}^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

If we use the distance traveled along the curve, $S(t)$ to denote position, where

$$S(t) = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

then the unit tangent vector can be specified independent of the length of the tangent vector, $|T|$.

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{d\vec{R}/dt}{|d\vec{R}/dt|} = \frac{d\vec{R}/dt}{dS/dt} = \frac{d\vec{R}}{dS}$$

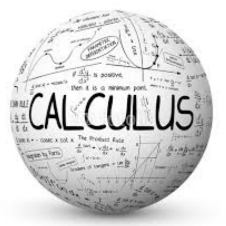
$$\hat{T} = \frac{d\vec{R}}{dS} = \frac{dx}{dS}\hat{i} + \frac{dy}{dS}\hat{j} + \frac{dz}{dS}\hat{k}$$

Example 1.4

Find the length of the curve of intersection between

$$\left.\begin{array}{lll} \text{surface 1} & \boldsymbol{y = 2x} & \text{(a plane)} \\ \text{surface 2} & \boldsymbol{z = \dfrac{x^2+y^2}{5}} & \end{array}\right\} \text{ for } \boldsymbol{z \le 25}$$

Curvature and Centripetal Acceleration



Reading

Trim 11.12 ⟶ *Normal Vectors, Curves, and Radius of Curvature*

11.13 ⟶ *Displacement, Velocity, and Acceleration*

Assignment

web page ⟶ *assignment #3*

Normal Vector

The normal to a point is the line which is perpendicular to the tangent vector, $\boldsymbol{T}$, given in the previous section. If the unit tangent vector in 2D space is given in terms of the length along the curve, $\boldsymbol{S}$

$$\hat{\boldsymbol{T}} = \frac{d\vec{\boldsymbol{R}}}{d\boldsymbol{S}} = \frac{d\boldsymbol{x}}{d\boldsymbol{S}}\hat{\boldsymbol{i}} + \frac{d\boldsymbol{y}}{d\boldsymbol{S}}\hat{\boldsymbol{j}}$$

then the unit normal vector, $\hat{\boldsymbol{N}}$, is

$$\hat{\boldsymbol{N}} = -\frac{d\boldsymbol{y}}{d\boldsymbol{S}}\hat{\boldsymbol{i}} + \frac{d\boldsymbol{x}}{d\boldsymbol{S}}\hat{\boldsymbol{j}}$$

Because of orthogonality $\hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{N}} = \boldsymbol{0}$. In 3D space there is not a single normal vector but an entire plane of normal vectors.

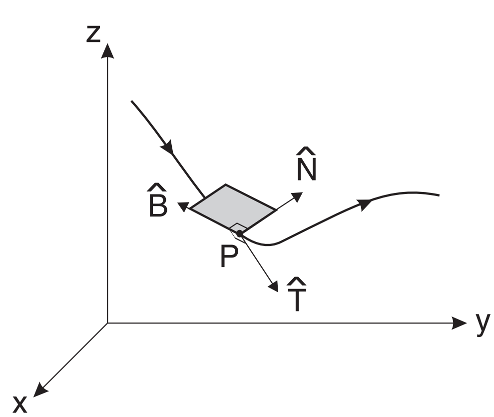
Two unique vectors can be identified in this plane of vectors, they are the *principal normal*, given as

$$\hat{\boldsymbol{N}} = \frac{\vec{\boldsymbol{N}}}{|\vec{\boldsymbol{N}}|} = \frac{d\hat{\boldsymbol{T}}/d\boldsymbol{t} \cdot (d\boldsymbol{t}/d\boldsymbol{S})}{\left|d\hat{\boldsymbol{T}}/d\boldsymbol{t} \cdot (d\boldsymbol{t}/d\boldsymbol{S})\right|} = \frac{d\hat{\boldsymbol{T}}/d\boldsymbol{S}}{\left|d\hat{\boldsymbol{T}}/d\boldsymbol{S}\right|}$$

and the *binormal* given as

$$\hat{\boldsymbol{B}} = \hat{\boldsymbol{T}} \times \hat{\boldsymbol{N}}$$

These vectors are shown in the following figure.



Curvature and Radius of Curvature

The rate of change of the unit tangent vector with respect to the distance travelled along the curve, i.e. $d\hat{T}/dS$ can be thought of as a measure of the rate of change of direction of $\hat{T}$ or a measure of the curvature of the curve.

Since the vector $\hat{T}$ has both direction and magnitude, the curvature of a curve is more aptly defined in terms of a curvature parameter, $\kappa(S)$, where

$$\kappa(S) = \text{rate of change of the unit tangent with respect to distance travelled}$$

$$\kappa(S) = \left|\frac{d\hat{T}}{dS}\right| = \left|\frac{d\hat{T}/dt}{dS/dt}\right| \qquad (1)$$

but since

$$\frac{dS}{dt} = \left|\frac{d\vec{R}(t)}{dt}\right| = \left|\vec{V}\right| \qquad (2)$$

and [from page 32 of the notes]

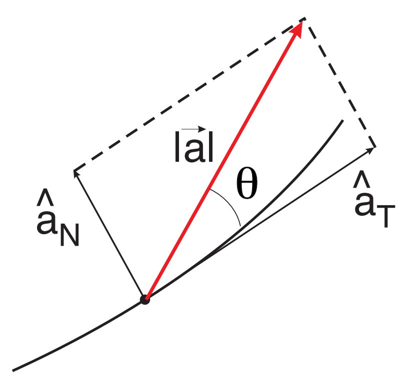
$$|\vec{a}_{\hat{N}}| = \left|\vec{V}\right|\left|\frac{d\hat{T}}{dt}\right| \qquad \Rightarrow \qquad \left|\frac{d\hat{T}}{dt}\right| = \frac{|\vec{a}_{\hat{N}}|}{\left|\vec{V}\right|} \qquad (3)$$

If we substitute Eqs. 2 and 3 into 1, we get

$$\kappa(S) = \frac{|\vec{a}_{\hat{N}}|}{\left|\vec{V}\right|} \cdot \frac{1}{\left|\vec{V}\right|}$$

Noting that

$$|\vec{a}_{\hat{N}}| = \frac{|\vec{V} \times \vec{a}|}{|\vec{V}|}$$



$$\kappa(S) = \frac{|\vec{V} \times \vec{a}|}{|\vec{V}|^3} \qquad \Rightarrow \qquad \boxed{\kappa(S) = \frac{|\vec{R}' \times \vec{R}''|}{|\vec{R}'|^3}} \qquad \text{and} \qquad \boxed{\rho(S) = \frac{1}{\kappa(S)}}$$

Centripetal Acceleration

Recall that the 2nd derivative (acceleration) can be written as

$$\begin{aligned}\frac{d^2\vec{R}}{dt^2} &= \hat{i}\frac{d^2x}{dt^2} + \hat{j}\frac{d^2y}{dt^2} + \hat{k}\frac{d^2z}{dt^2} \\ &= \underbrace{\frac{d\left|\vec{V}\right|}{dt}\hat{T}}_{tangent\ to\ curve} + \underbrace{\left|\vec{V}\right|\left|\frac{d\hat{T}}{dt}\right|\hat{N}}_{normal\ to\ curve}\end{aligned}$$

where the acceleration normal to the path is given as

$$|\vec{a}_{\hat{N}}| = \left|\vec{V}\right|^2 \kappa = |\vec{V}|^2/\rho$$

Therefore

$$\frac{d^2\vec{R}}{dt^2} = \frac{d\left|\vec{V}\right|}{dt}\hat{T} + \frac{\left|\vec{V}\right|^2}{\rho}\hat{N}$$

This clearly relates the centripetal acceleration locally to the curvature of path followed by the particle.

The limiting case can be shown by looking at the motion along a straight line path

$$\rho \to \infty \quad \text{(everywhere)}$$

$$\frac{d^2\vec{R}}{dt^2} = \frac{d\left|\vec{V}\right|}{dt}\hat{T}$$

since the only acceleration is along the path.

For a circular path of radius R, with a particle moving at a constant speed $\vec{V}_0$

$$\frac{d^2\vec{R}}{dt^2} = \frac{\left|\vec{V}\right|^2}{\rho}\hat{N} = \frac{\left|\vec{V}\right|^2}{R}\hat{N}$$

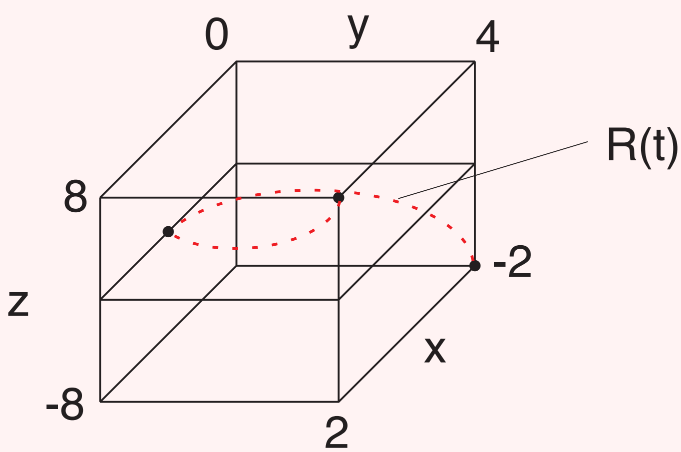
The only acceleration is perpendicular to the path.

Example 1.5

A particle moves through $3-$D space so that its position vector at time t is

$$\vec{R}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

The path is referred to as a twisted cube as shown below



Find the tangential and normal components of acceleration at time t.

Example 1.6: This example gives a good summary of the types of vector/3D curve calculations you should be able to do after working through chapter 11 of Trim.

A particle follows a trajectory in space given by

$$\begin{aligned} x(t) &= 2\cos t \\ y(t) &= 2\sin t \\ z(t) &= 2\pi - t \end{aligned}$$

$$\boxed{\begin{aligned} &x^2 = 4\cos^2 t \\ &y^2 = 4\sin^2 t \\ &x^2 + y^2 = 4(\cos^2 t + \sin^2 t) = 4 \end{aligned}}$$

with x, y, z in meters and $0 \le t \le 2\pi$ in seconds.

a) sketch the trajectory curve

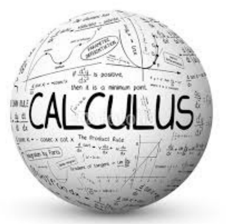
b) calculate local coordinates $\hat{T}, \hat{N}, \hat{B}$ to the curve at any time t

c) calculate the length of the curve

d) find the curvature at any time t

e) express the particle velocity and acceleration at any time t in $\hat{T}, \hat{N}$ coordinates

Application of Vector Products: Force Analysis - Backing Up A Trailer



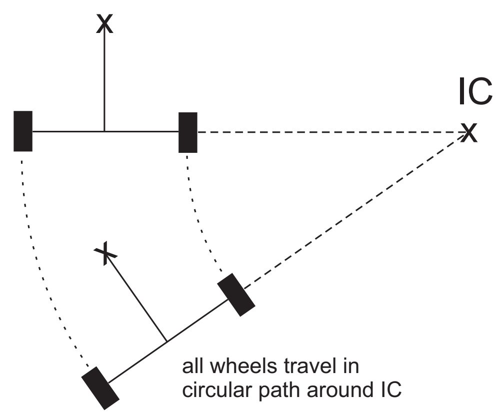
Reading
Trim 11.7 $\longrightarrow$ *Physical Applications of Scalar and Vector Products*

Assignment
web page $\longrightarrow$ *project #1*

Modelling Concepts

Instantaneous Center: an imaginary point that is the center of rotation of the vehicle, trailer

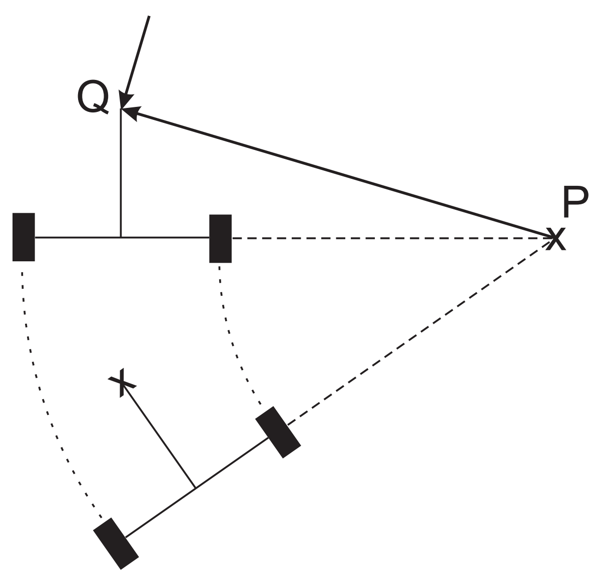
- where all wheels travel in a circular path around the IC



Applying Force:

- at the hitch

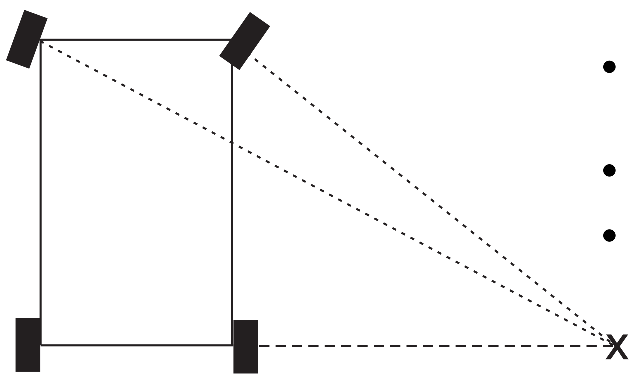
- moment at point $\boldsymbol{Q}$ about point $\boldsymbol{P}$
- $\vec{\boldsymbol{F}}$ must be $\perp$ $\vec{\boldsymbol{PQ}}$
- why? $\Rightarrow$ motion is <u>rotational</u> only - not translational



Project

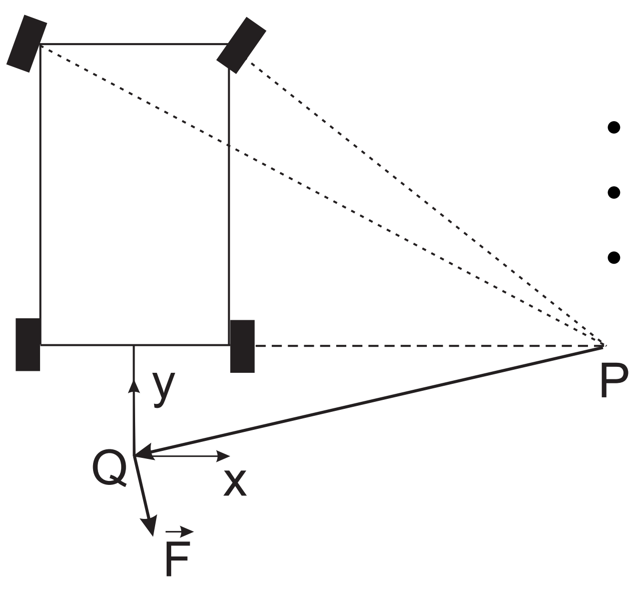
Project Goal #1: Backing up the trailer "by hand" is easy (relatively) to understand - adding a vehicle becomes more complicated.

Vehicle has instantaneous center



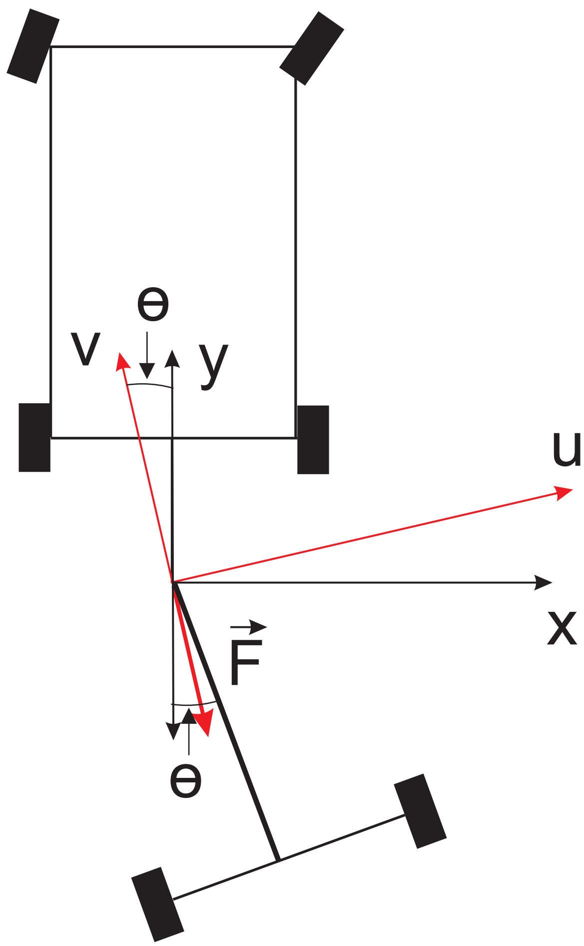
- imaginary point that is the center of rotation of the vehicle
- all wheels trace a circular path around IC
- acts as a point of rotation of moment vector at trailer hitch

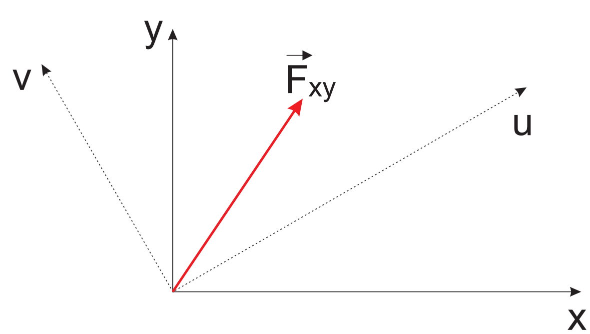
Vehicle moment transmits force to trailer



- same as trailer
- $\vec{F} \perp \vec{PQ}$ (IC to hitch)
- trailer response to force depends on position of trailer relative to the vehicle

- set up coordinate system transformation from $\vec{F}_{xy}$ (vehicle) to $\vec{F}_{uv}$ (trailer)





recall: $\vec{u} \cdot \vec{v} = u_v$

$\Rightarrow$ component of $\vec{u}$ in $\hat{v}$ direction

$$\begin{aligned} F_u &= \vec{F}_{xy} \cdot \hat{u} \\ F_v &= \vec{F}_{xy} \cdot \hat{v} \end{aligned}$$

We can use the same approach to define a local coordinate system based on (u, v)

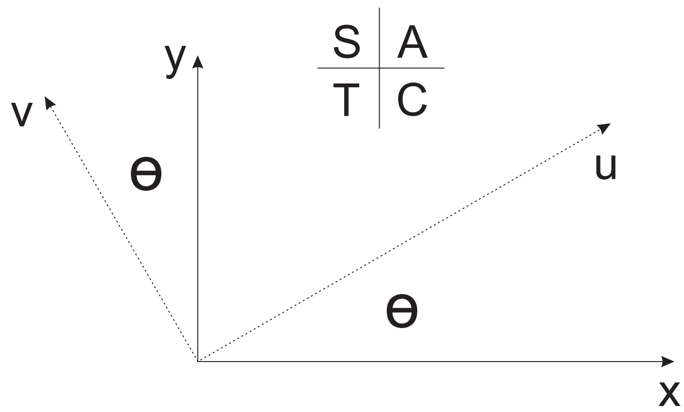
$$F_u = \vec{F}_{xy} \cdot \hat{i}_u \qquad \leftarrow \text{special unit vector, } (1,0,0)\text{, based on local } (u,v) \text{ coords}$$

$$\begin{aligned} F_u &= (F_x \hat{i}_x + F_y \hat{j}_y) \cdot \hat{i}_u \\ F_y &= (F_x \hat{i}_x + F_y \hat{j}_y) \cdot \hat{j}_v \end{aligned}$$

or expressed in matrix form

$$\begin{bmatrix} \boldsymbol{F_u} \\ \boldsymbol{F_v} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{i}}_x \cdot \hat{\boldsymbol{i}}_u & \hat{\boldsymbol{j}}_y \cdot \hat{\boldsymbol{i}}_u \\ \hat{\boldsymbol{i}}_x \cdot \hat{\boldsymbol{j}}_v & \hat{\boldsymbol{j}}_y \cdot \hat{\boldsymbol{j}}_v \end{bmatrix} \begin{bmatrix} \boldsymbol{F_x} \\ \boldsymbol{F_y} \end{bmatrix}$$

$\vec{\boldsymbol{F}}_{uv} = \boldsymbol{R}\, \vec{\boldsymbol{F}}_{xy}$ where $\boldsymbol{R}$ = transformation matrix



$$\begin{aligned}
\hat{\boldsymbol{i}}_x \cdot \hat{\boldsymbol{i}}_u &= \left|\hat{\boldsymbol{i}}_x\right| \left|\hat{\boldsymbol{i}}_u\right| \cos\theta = \cos\theta \\
\hat{\boldsymbol{j}}_x \cdot \hat{\boldsymbol{i}}_u &= \cos(\pi/2 - \theta) = \sin\theta \\
\hat{\boldsymbol{i}}_x \cdot \hat{\boldsymbol{j}}_v &= \cos(\pi/2 + \theta) = -\sin\theta \\
\hat{\boldsymbol{j}}_y \cdot \hat{\boldsymbol{j}}_v &= \cos\theta
\end{aligned}$$

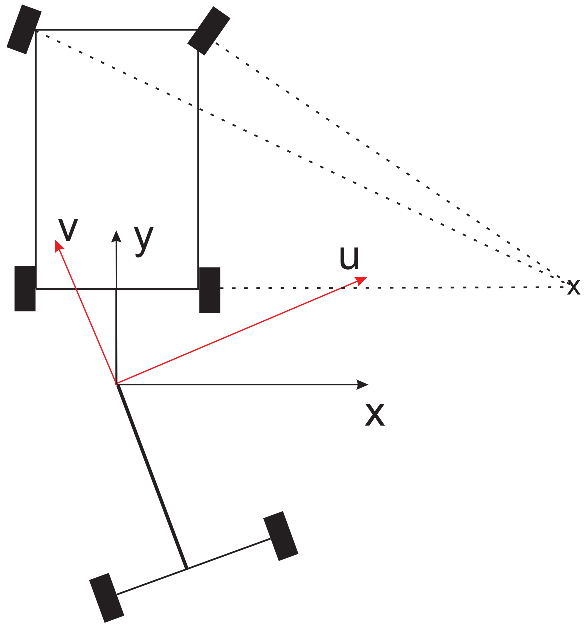
Therefore

$$\begin{bmatrix} \boldsymbol{F_u} \\ \boldsymbol{F_v} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{F_x} \\ \boldsymbol{F_y} \end{bmatrix}$$

Example

Find coordinates of vector $\vec{\boldsymbol{F}}_{xy}\ (3, 1, 0)$ in $\boldsymbol{u}\ \boldsymbol{v}$ coordinates for angle of rotation of $\theta = \pi/9$

Project Goal #2: Develop model predictions, verify that the vehicle/trailer behaves as predicted



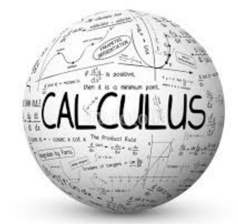
ANALYSIS ← must do this first
IC vehicle given for fixed steering positions
→ show car

1. calculate moment $\vec{F}_{xy}$
2. transform to $\vec{F}_{uv}$
3. find IC for trailer, path travelled by wheels

Measure → verification of IC, track

Differential Calculus of Multivariable Functions

Introduction to Multivariable Functions



Reading
Trim 12.1 $\longrightarrow$ *Multivariable Functions*
12.2 $\longrightarrow$ *Limits and Continuity*

Assignment
web page $\longrightarrow$ *assignment #3*

Functions of Two Independent Variables

If z is uniquely determined when the values of x and y are given, we say that z is a single-valued function of x and y and indicate this fact by the notation

$$z = f(x, y)$$

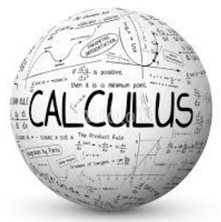
Many engineering applications exist that show this functional dependence, for instance the vibration of a string or the temperature of a plate or a cooling fin

$$T = f(x, y)$$

It is sometimes useful to visualize the function graphically. The function $z = f(x, y)$ can be visualized in two different ways:

1. a contour plot
2. a surface plot

Partial Derivatives



Reading

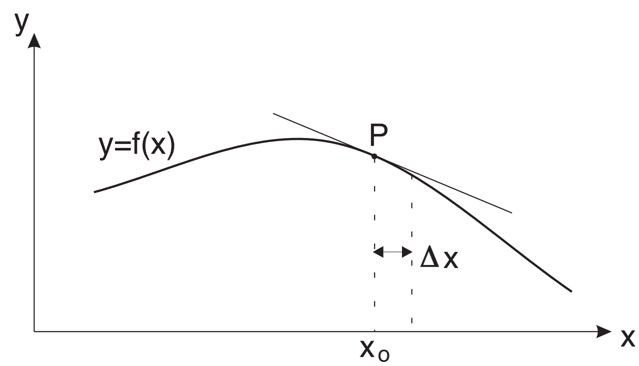
Trim 12.3 $\longrightarrow$ *Partial Derivatives*

12.5 $\longrightarrow$ *Higher Order Partial Derivatives*

Assignment

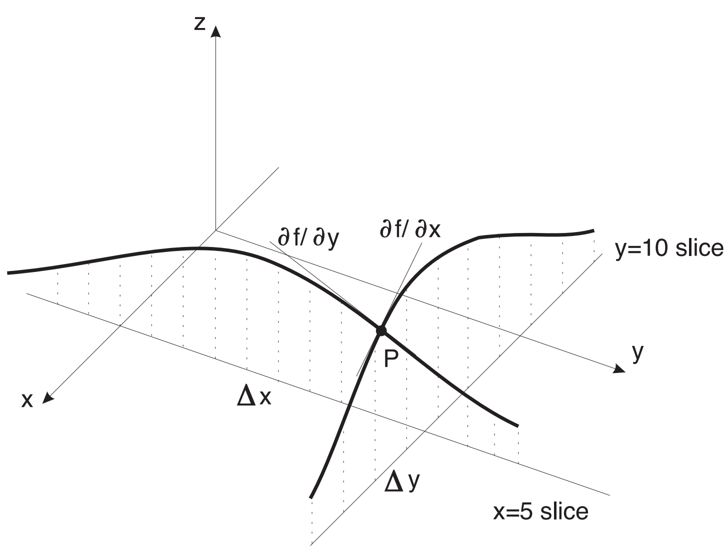
web page $\longrightarrow$ *assignment #4*

For $y = f(x)$ there is only one slope at any point P, where the slope of the tangent at P is



$$slope = \left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \left.\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\right|_{at\ x=x_0}$$

However for a 3-dimensional surface, $z = f(x, y)$, the slope varies depending on the direction you move on the surface.



For $\quad z(x, y) = 20e^{\left(-x^2/100\right)} \cos \dfrac{\pi y}{30} \quad$ find the slope at $P(x = 5, y = 10)$

Different notations of the partial derivative can be used.

$$\frac{\partial f}{\partial x} \quad \text{or} \quad f_x \qquad\qquad \frac{\partial f}{\partial y} \quad \text{or} \quad f_y$$

Higher Orders

$$\frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad f_{yx} = f_{xy}$$

$\longleftarrow$ order of calculation $\qquad$ $\longrightarrow$ order of calculation

Higher Order Partial Derivatives

Partial derivatives of the second order are denoted as

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

where

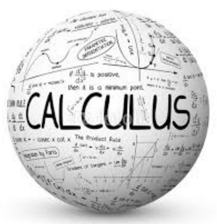
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

as so on.

If the function $z = f(x, y)$ together with the partial derivatives f_x, f_y, f_{xy} and f_{yx} are continuous, then

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Chain Rule for Partial Derivatives



`Reading`
Trim 12.6 $\longrightarrow$ *Chain Rules for Partial Derivatives*

`Assignment`
web page $\longrightarrow$ *assignment #4*

Review of Single Variable Functions

- if we understand for 1 variable, it is easy to understand for 2 or 3 variables
- the changes are straight forward

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Using the tree we need to remember $\Rightarrow$

$$\begin{array}{c} y \\ | \\ u \\ | \\ x \end{array}$$

$LHS \rightarrow \dfrac{dy}{dx} \rightarrow$ implies we are looking at the function $y(x)$

$RHS \rightarrow \dfrac{dy}{du} \rightarrow$ implies we are looking at the same y but now the independent variable is u, i.e. $y(u)$.

The chain rule relates slopes of the 2 plots

$$\underbrace{\frac{dy(x)}{dx}}_{slope\ @\ any\ point\ x} = \underbrace{\frac{dy(u)}{du}}_{\lim_{u\to 0}\frac{y(u+\Delta u)-y(u)}{\Delta u}} \times \underbrace{\frac{du(x)}{dx}}_{the\ effect\ of\ a\ change\ of\ scale}$$

Chain Rule for Multivariable Functions

1. Given a dependent variable as a function of several independent variables, transform the dependent variable into a new set of independent variables:

$$\underbrace{z}_{dependent} \quad \overbrace{(u,v)}^{\rightarrow\, independent} \quad \longrightarrow \quad z(x,y)$$

   You will typically be asked to find

$$\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial y^2}$$

2. map the dependency of the dependent variable in terms of the independent variables, in our case, $z(u,v)$

```
  z
 / \
u   v
```

3. map the dependency of the independent variables in the transformed space in our case, $u(x,y)$ and $v(x,y)$

```
        z
      /    \
   u          v
  / \        / \
 x   y      x   y
```

4. use the chain rule to form all paths from $z \rightarrow x$ and $z \rightarrow y$

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}
\end{aligned}$$

5. How do we find the 2nd derivative?

   - remember $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x} \underbrace{\left(\dfrac{\partial z}{\partial x}\right)}_{\text{we already know how to do this}}$

- replace z at the top of the tree with $\dfrac{\partial z}{\partial x}$

$$
\begin{array}{ccccccc}
 & & & \partial z/\partial x & & & \\
 & & / & & \backslash & & \\
 & u & & & & v & \\
/ & & \backslash & & / & & \backslash \\
x & & y & & x & & y
\end{array}
$$

- find all paths between $\partial z/\partial x$
- remember, we may need to do the same for y if asked

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\right]\frac{\partial u}{\partial x} + \left[\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\right]\frac{\partial v}{\partial x} \\
\frac{\partial^2 z}{\partial y^2} &= \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\right]\frac{\partial u}{\partial y} + \left[\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\right)\right]\frac{\partial v}{\partial y}
\end{aligned}
$$

---

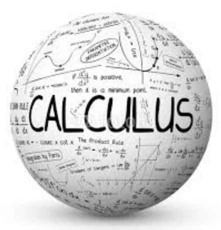
for 2nd derivatives, let

$$
\frac{\partial z}{\partial x} = A, \qquad \frac{\partial z}{\partial y} = B
$$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x^2} = \frac{\partial A}{\partial x} &= \frac{\partial A}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial A}{\partial v}\frac{\partial v}{\partial x} \\
&= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial y^2} = \frac{\partial B}{\partial y} &= \frac{\partial B}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial B}{\partial v}\frac{\partial v}{\partial y} \\
&= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right)
\end{aligned}
$$

Gradients



Reading
Trim 12.4 $\longrightarrow$ *Gradients*

Assignment
web page $\longrightarrow$ *assignment #4*

At each point on a surface $z = f(x, y)$ we can define two slopes - one in each spatial direction.

From the previous example: $z(x, y) = 20e^{-x^2/100}\ \cos \pi y/30$, where

at $P_{x=5,y=10}$ $\quad \dfrac{\partial f}{\partial x} = -0.78 \quad \dfrac{\partial f}{\partial y} = -1.41.$

It is useful to combine these into one vector measure of the slope of the surface at P. This is called the **Gradient Vector**.

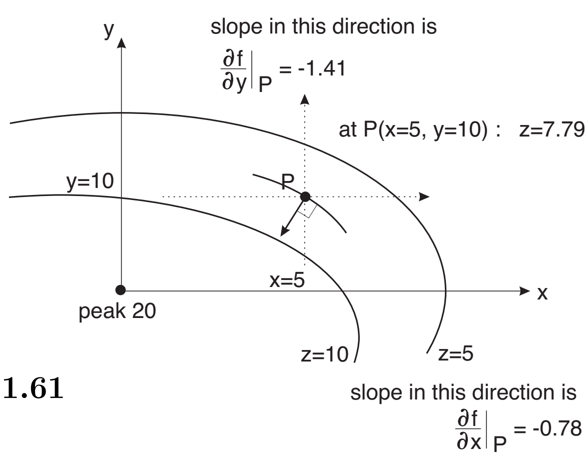
$$\boxed{\text{grad } f = \nabla f \equiv \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}}$$

where the "del operator" - ∇ is defined in 2D as: $\quad \nabla = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y}$

$$\text{grad } f\Big|_P = \nabla f\Big|_P = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} = \hat{i}(-0.78) + \hat{j}(-1.41)$$

$$f = z = 20e^{-x^2/100} \cos\left(\frac{\pi y}{30}\right)$$

The meaning is clear on a contour plot



The magnitude of the gradient vector is

$$|\nabla f_{atP}| = \sqrt{(-1.41)^2 + (-0.78)^2} = 1.61$$

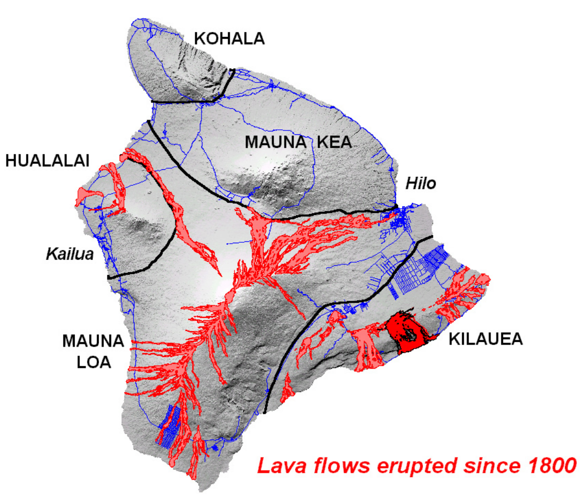
The gradient vector, $\boldsymbol{\nabla f}$ is:

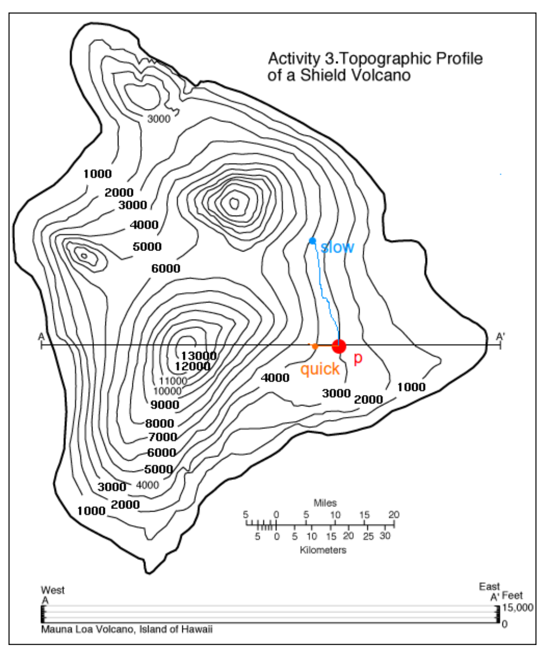
- a vector at $\boldsymbol{P}$ that is **perpendicular** to the contour at $\boldsymbol{P}$
- **points uphill** in the direction in which slope increases the most at $\boldsymbol{P}$
- its magnitude is $|\boldsymbol{\nabla f_{atP}}| = \mathbf{1.61}$. If we move $\mathbf{1}\ \boldsymbol{unit}$ in the $\boldsymbol{\nabla f}$ direction, then $\boldsymbol{z}$ increases by $\mathbf{1.61}\ \boldsymbol{units}$. This is the biggest slope in any direction at $\boldsymbol{P}$.

The gradient comes up in many applications because things tend to move down the gradient.

Examples:

1. If we release a ball at point $\boldsymbol{P}$ from rest it will roll down the gradient
   - the direction will be in the negative $\boldsymbol{\nabla f}$ direction
   - how far it rolls will depend on the magnitude $|\boldsymbol{\nabla f_{atP}}|$
2. Heat flow in a cooling fin
   - in the cross section of the fin, $\boldsymbol{T = f(x, y)}$
   - the heat flow at some point $\boldsymbol{P}$ is in the $\boldsymbol{\nabla f}$ direction
   - how much heat depends on the magnitude $|\boldsymbol{\nabla f_{atP}}|$
   - Fourier's law in 2D is $\boldsymbol{\vec{q} = -k\nabla T}$
3. Lava flowing from a volcano
   - Mauna Loa and Kilauea on the Big Island of Hawaii are highly active
   - the gradient can be used to predict to path of lava flowing to the sea





Functions of 3 or More Independent Variables

The procedure can be extended to any number of independent variables. In Mechanical Engineering, the biggest number is usually 4. You may have more in abstract applications.

Examples: Solving for the transient temperature response in a room

$$T = f(x, y, z, t)$$

Includes 3 spatial variables and 1 time variable. Graphical representation becomes difficult.

Possible graphical representations include:

- multiple contour plots
- level surfaces (surfaces of constant value)
- "slice" - show T vs. z and fixed locations (x_0, y_0). Need many "slices" for full representation

Gradient in 3-D

For both $T = f(x, y, z)$ and $T = f(x, y, z, t)$ the vector gradient is

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

with the del operator

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

since the gradient vector is a measure in space only. ∇f is a vector in 3-D space. If $T = f(x, y, z, t)$ then ∇f will change with time.

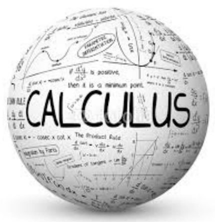
Example 2.1

Find the temperature variation inside a spherical lead shield enclosing a small radioactive sphere. The temperature will vary with radius. Suppose for example, that

$$T = f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

where $0.05 \leq x, y, z \leq 2$

Directional Derivatives



Reading
Trim 12.8 $\longrightarrow$ *Directional Derivatives*

Assignment
web page $\longrightarrow$ *assignment #4*

We know how to find a vector that points in the direction of the maximum and minimum change on slope but how do we account for the rate of change in slope in any other arbitrary direction. For this we introduce the directional derivative.

THEOREM: *let $f(x, y, z)$ be continuous and possess partial derivatives f_x, f_y, f_z throughout some neighborhood of the point $P_0(x_0, y_0, z_0)$. Let $f_x,\ f_y$ and f_z be continuous at P_0. Then the directional derivative at P_0 exists for a unit vector $\hat{v} = (v_x, v_y, v_z)$ in the direction of v, such that*

To find the gradient along a line segment s

$$\begin{aligned}\frac{df}{ds} &= \underbrace{\frac{df}{dx}}_{f_x}\underbrace{\frac{dx}{ds}}_{v_x} + \frac{df}{dy}\frac{dy}{ds} + \frac{df}{dz}\frac{dz}{ds} \\ &= f_x v_x + f_y v_y + f_z v_z\end{aligned}$$

where f is expressed in terms of s, where s is a measure of directed distance along a vector and the unit vector is $\hat{v} = (v_x, v_y, v_z)$. First set $s = 0$ at $P_0 = (x_0, y_0, z_0)$.

To find f in this transformed space, we can write

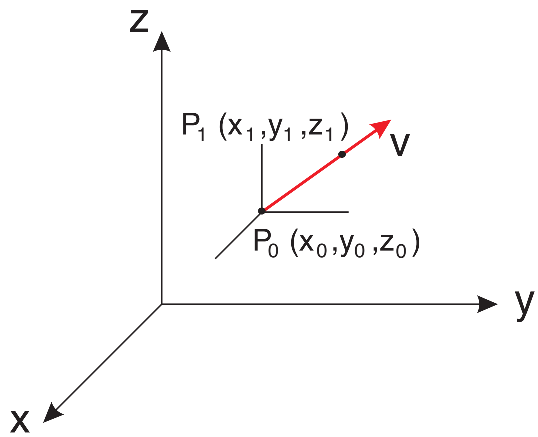
$$x = x_0 + v_x s$$

$$y = y_0 + v_y s$$

$$z = z_0 + v_z s$$

Note: we start at $P_0(x_0, y_0, z_0)$ and attach a vector in the $\hat{v}(v_x, v_y, v_z)$ direction with a magnitude of s

Then



$$
\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \\
&= \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y + \frac{\partial f}{\partial z} v_z \\
&= D_v f \\
&= \text{directional derivative in direction } v
\end{aligned}
$$

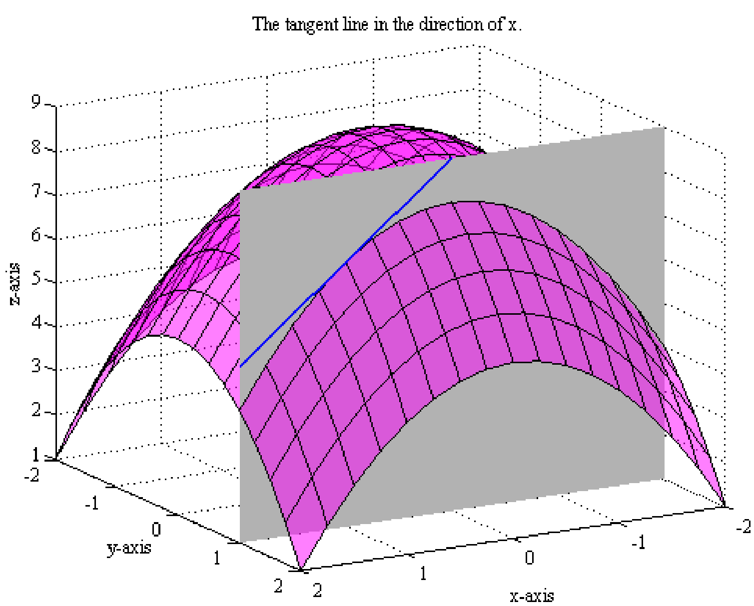
The directional derivative of f at P_0 in the direction v can also be defined as the average rate of change of f with respect to distance as P_1 approaches P_0 along s.

$$
\frac{\partial f}{\partial s} = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} = \lim_{P_1 \to P_0} \frac{f(x_1, y_1, z_1) - f(x_0, y_0, y_0)}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}}
$$

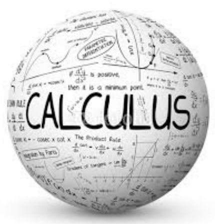
This can be expressed as a scalar product of the gradient of f and the vector v.

$$
D_v f = \nabla f \cdot \hat{v}
$$

this can be thought of as the projection of ∇f onto the vector v.



Tangent Lines and Tangent Planes



Reading
Trim 12.9 $\longrightarrow$ *Tangent Lines and Tangent Planes*

Assignment
web page $\longrightarrow$ *assignment #4*

This is another application of the gradient.

The basic idea is that the ∇f is perpendicular to a contour line in 2D and perpendicular to a level surface in 3D. Therefore the ∇f vector will be perpendicular to a tangent line or plane. We will use this to find the tangent line or plane at a point.

$$z = 20e^{-x^2/100}\cos\left(\frac{\pi y}{30}\right)$$

Find the plane tangent to the surface at point $P(x = 5,\ y = 10)$.

one way is to use the Taylor series, keeping only the linear terms

$$z = 7.79 - 0.78(x-5) - 1.41(y-10)$$

another way is to use the gradient ideas described above. We can write the equation of the hill as

$$F(x, y, z) = 20e^{-x^2/100}\cos\left(\frac{\pi y}{30}\right) - z$$

$\hookrightarrow$ $\boxed{\begin{array}{l}\text{move } z \text{ to the righthand side}\\ \text{if } F = 0 \Rightarrow \text{on surface}\\ \text{if } F < 0 \Rightarrow \text{outside}\\ \text{if } F > \Rightarrow \text{inside}\end{array}}$

In 3D the level surface $F = 0$ defines the hill surface

$$\begin{aligned}
\nabla F &= \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}\\
&= \hat{i}\left[20e^{-x^2/100}\left(-\frac{2x}{100}\right)\cos\left(\frac{\pi y}{30}\right)\right]\\
&\quad + \hat{j}\left[20e^{-x^2/100} - \left(\frac{\pi}{30}\right)\sin\left(\frac{\pi y}{30}\right)\right] + \hat{k}(-1)
\end{aligned}$$

at $\boldsymbol{P(5, 10, 7.79)}$

$$\boldsymbol{\nabla F|_{at\ P} = \hat{\imath}(-0.78) + \hat{\jmath}(-1.41) + \hat{k}(-1)}$$

$\boldsymbol{\nabla F|_{at\ P}}$ is perpendicular to the $\boldsymbol{F = 0}$ surface and also perpendicular to a tangent plane at $\boldsymbol{P}$.

We have:

- a point on the plane $\boldsymbol{P(5, 10, 7.79)}$
- a vector normal to the plane $\boldsymbol{\vec{N} = \nabla F}$

Therefore the equation of the tangent plane is

$$\begin{aligned} N_x(x - x_P) + N_y(y - y_P) + N_z(z - z_P) &= 0 \\ -0.78(x - 5) - 1.41(y - 10) - 1(z - 7.79) &= 0 \end{aligned}$$

$$\boldsymbol{z = 7.79 - 0.78(x - 5) - 1.41(y - 10)}$$

This is the same as the Taylor series.

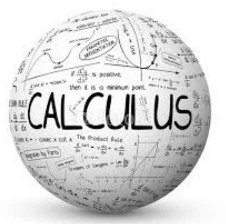
$$\boxed{\textbf{Tangent Plane :}\ \ A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

$$\boxed{\textbf{Normal Line :}\ \ \frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}}$$

where

$$\boxed{A = f_x(x_0, y_0), \quad B = f_y(x_0, y_0), \quad C = -1}$$

Extrema of Functions

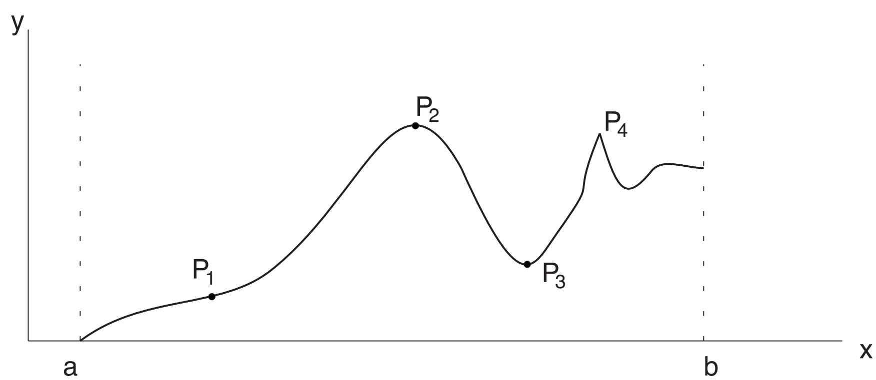


Reading
Trim 12.10 $\longrightarrow$ *Relative Maxima and Minima*
12.11 $\longrightarrow$ *Absolute Maxima and Minima*

Assignment
web page $\longrightarrow$ *assignment #5*

Review of Functions of One Variable

The critical points are identified by the x-values where $f' \to 0$ or where f' does not exist. We apply the 2nd derivative test at these critical points:



1. at P_1: $f' = 0$ and $f'' = 0$
2. at P_2: $f' = 0$ and $f'' < 0$
3. at P_3: $f' = 0$ and $f'' > 0$
4. at P_4: f' and f'' do no exist, i.e. $\to \infty$

The <u>absolute</u> max/min in the range $a \to b$ is either at one of the critical points or at the end points of the domain, Note: end points must be checked.

- we can see from the figure, that the absolute minimum in the range $a \to b$ is at $f(a)$ and that the absolute maximum in the range $a \to b$ is at $x = P_2$
- if the test fails, we can always plot up $y = f(x)$ to see what it looks like
- sometimes it is easier to use plots or an understanding of the physics associated with the problem to decide rather that formal tests

Functions of Two Variables

We will examine $z = f(x, y)$ where $\nabla f = 0$, i.e. the "flat" portion of the curved surface. The critical points are where

$$\frac{\partial f}{\partial x} = 0 \qquad\qquad \frac{\partial f}{\partial y} = 0$$

In this instance, the 2nd derivative tests involve $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$.

Possible max/min locations include:

1. Peak Shape: Relative maximum at $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$

   $$\frac{\partial^2 f}{\partial x^2} < 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} < 0$$

   The $x-$ and $y-$slopes are decreasing +ve to -ve

2. Bowl Shape: Relative minimum at $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$

   $$\frac{\partial^2 f}{\partial x^2} > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} > 0$$

   The $x-$ and $y-$slopes are increasing -ve to +ve

3. Saddle Shape: "saddle point" at $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$, there is no clear min or max at $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$. It looks like a max in x and a min in y.

   $$\frac{\partial^2 f}{\partial x^2} < 0 \quad \text{but} \quad \frac{\partial^2 f}{\partial y^2} > 0$$

4. Bowl Shape on x and y axes, but decreasing in the s direction

   This is where the $\dfrac{\partial^2 f}{\partial x \partial y}$ derivative comes in.

If the critical point is at P and

$$\left.\frac{\partial f}{\partial x}\right|_P = 0, \quad \left.\frac{\partial f}{\partial y}\right|_P = 0,$$

or either or both are undefined at P, then compute

$$\begin{aligned} A &= \left.\frac{\partial^2 f}{\partial x^2}\right|_P \\ B &= \left.\frac{\partial^2 f}{\partial x \partial y}\right|_P \\ C &= \left.\frac{\partial^2 f}{\partial y^2}\right|_P \\ D &= B^2 - AC \end{aligned}$$

if	$D < 0$	and	$A < 0$	$\Rightarrow$	relative max at P
if	$D < 0$	and	$A > 0$	$\Rightarrow$	relative min at P
if	$D > 0$			$\Rightarrow$	saddle point at P
if	$D = 0$			$\Rightarrow$	test fails - plot to examine

To find the absolute max/min in the interval:

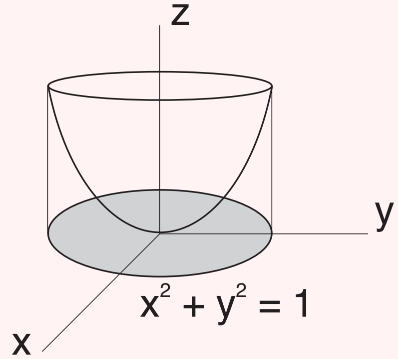
- check $f(x, y)$ values at all critical points and on all the boundary points i.e. edges of the interval
- the boundary points are more of an issue in 2D than the 1D case

Example 2.2

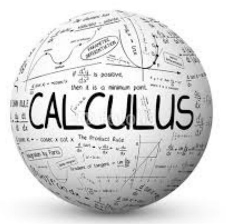
Find the max. or min. point for the function

$$z = f(x, y) = x^2 + xy + y^2$$

in the domain $x^2 + y^2 \leq 1$.



Method of Least Squares



Reading
Trim 12.13 $\longrightarrow$ *Least Squares*

Assignment
web page $\longrightarrow$ *assignment #6*

We will demonstrate the method through example.

Given four data points from an experiment

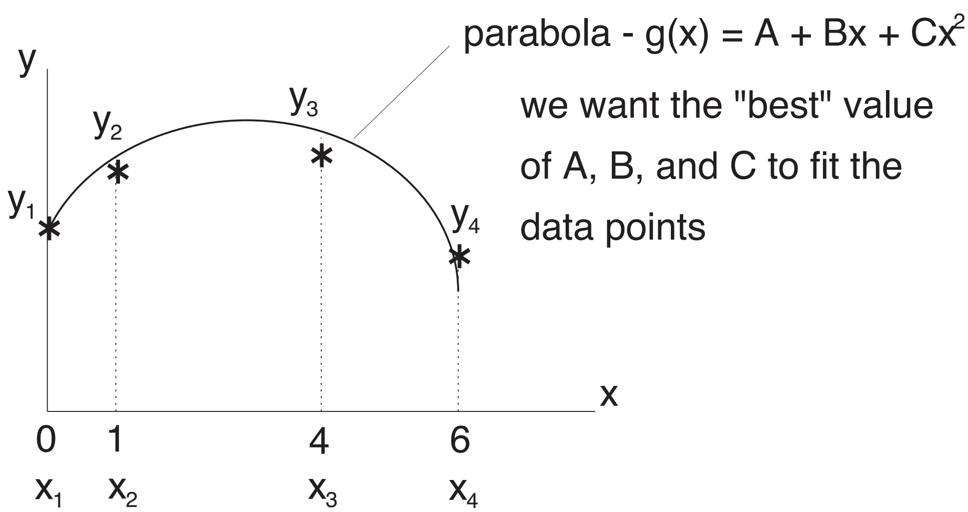
x_i	0	1	4	6
y_i	2	3	3	1

where $N = 4$ is the number of data points.

Fit the "best" quadratic curve through the data.

$$g(x) = A + Bx + Cx^2$$

The curve can take any form we want. Often, these is some physical basis to guide us in determining which form of the curve to use. Sometimes we might use trial and error for a few different curve shapes.



The question remains - how good is the fit? At each x_i data point, we must compute the difference (residual). At $\boldsymbol{x_i}$

$$\boldsymbol{R_i} = \underbrace{\boldsymbol{g(x_i)}}_{\textit{curve fit value at } x_i} - \underbrace{\boldsymbol{y_i}}_{\textit{data value at } x_i}$$

The "best" curve would have minimum $\boldsymbol{R_i}$ summed over all data points. However, some $\boldsymbol{R_i}$'s are +'ve and some are -'ve. The cancellation effect can be misleading, giving a false indication of a good fit. To avoid this, we need to square $\boldsymbol{R_i}$, so all are +'ve. Therefore the "best" curve is based on the minimum $\boldsymbol{R_i^2}$, summed over all points.

The sum of the square of the residuals is given as

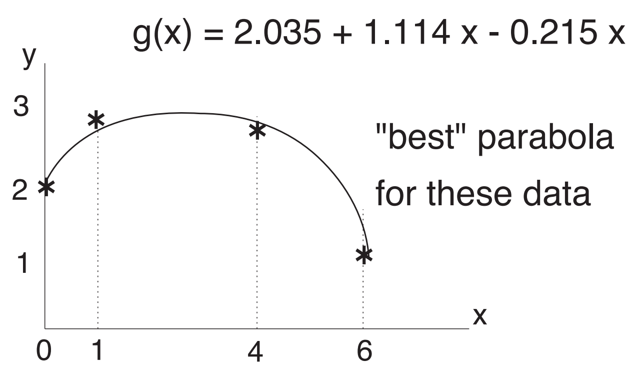
$$\boldsymbol{S} = \sum_{i=1}^{N} [\boldsymbol{g(x_i) - y_i}]^2 = \sum_{i=1}^{N} \left[\boldsymbol{A + Bx_i + Cx_i^2 - y_i}\right]^2$$

$\boldsymbol{x_i}$ and $\boldsymbol{y_i}$ are the given data points. The unknowns here are $\boldsymbol{A, B, C}$. i.e. $\boldsymbol{S = S(A, B, C)}$. The best fit will have the minimum value of $\boldsymbol{S}$. For minimum $\boldsymbol{S}$

$$\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{A}} = 0 \qquad \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{B}} = 0 \qquad \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} = 0$$

We need to solve three equations to find the best values for our three unknowns, $\boldsymbol{A, B, C}$.

The physical meaning of this is - on average, a data point is 0.0464 away from the $\boldsymbol{g(x)}$ curve - which is considered a good fit. (similar to a RMS type average).



The procedure can be used for any curve shape, however, it is sometimes convenient to change the variable first.

If we want to fit the data $\boldsymbol{x_i}$ and $\boldsymbol{y_i}$ to $\boldsymbol{g(x) = Ae^{Bx}}$, it is easier to take the $\ln$, i.e.

$$\textbf{data} \quad \boldsymbol{x_i}, \qquad \boldsymbol{Y_i = \ln y_i}$$

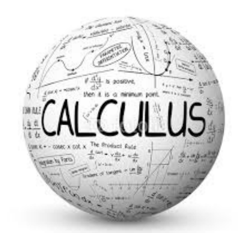
$$\ln \boldsymbol{g(x)} = \ln \boldsymbol{A + Bx}$$

$$\Rightarrow \quad \boldsymbol{G(x) = A^* + Bx}$$

We will fit a line $\boldsymbol{A^* + Bx}$ to $\boldsymbol{x_i, Y_i}$ data.

Multiple Integrals

Review of Single Integrals



Reading

Trim 7.1 $\longrightarrow$ *Review Application of Integrals: Area*

7.2 $\longrightarrow$ *Review Application of Integrals: Volumes*

7.3 $\longrightarrow$ *Review Application of Integrals: Lengths of Curves*

Assignment

web page $\longrightarrow$

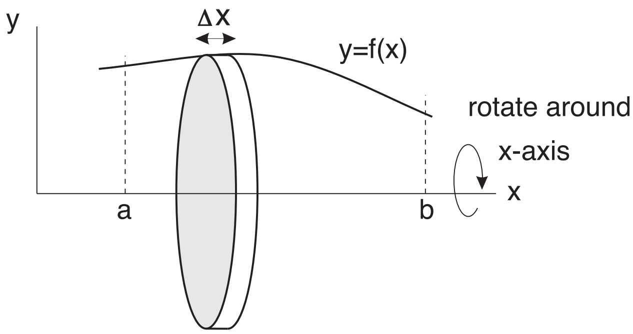
Planar Area

In the limit as $\Delta x \rightarrow dx$ the total number of panels $\rightarrow \infty$

$$A = \int_a^b y \cdot dx = \int_a^b f(x)dx$$

Volume of Solid of Revolution

a) **Disk Method** : rotate $y = f(x)$ about the $x-$ axis to form a solid.



The disk has a volume of $\mathcal{V} = \pi y^2 \Delta x$.

The total volume between a and b can be determined as:

$$\boxed{\mathcal{V} = \int_a^b \pi y^2 dx}$$

Note: The value of $y = f(x)$ is substituted into the formulation for area and the resulting equation is integrated between a and b.

b) **Shell Method:** Find a ring defined with ring area: $2\pi y \cdot \Delta y$.

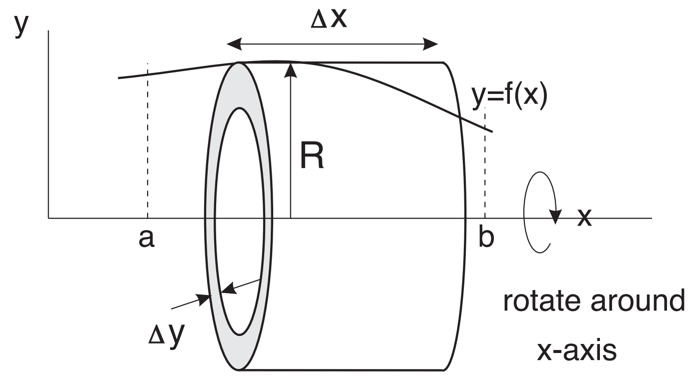
The volume of the ring is given by

$$\Delta\mathcal{V} = (2\pi y \cdot \Delta y)\Delta x$$

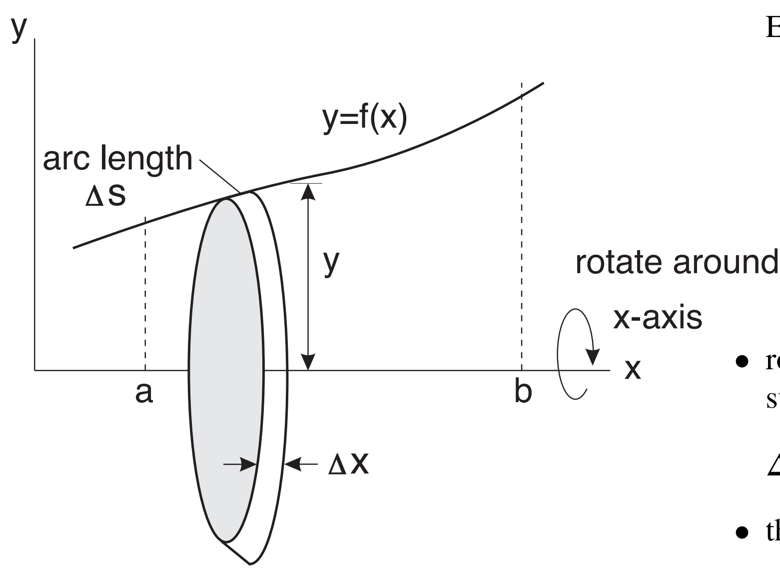
The volume of the solid is determined by solving the integral

$$\boxed{\mathcal{V} = \int_0^R 2\pi\, x\, y\, dy}$$

Either method can be used, which ever is most convenient.



Surface Area of Solid of Revolution



- the arc length can be defined using Eq. 7.15:

$$\Delta s = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\Delta x = \frac{\sqrt{dx^2+dy^2}}{dx}\Delta x$$

- rotate about the $x-$ axis, where the surface area is defined as

$$\Delta A_{surf} = 2\pi y \Delta s$$

- the total surface area is given as

$$\boxed{A_{surface} = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

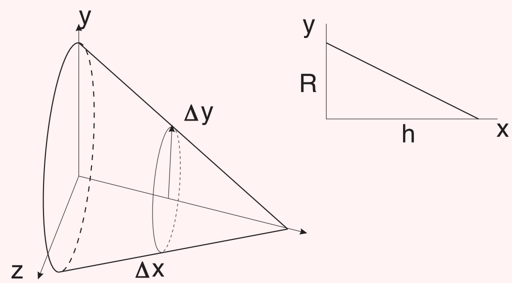
Example: 3.1

Find the area in the positive quadrant bounded by

$$y = \frac{1}{4}x \qquad \text{and} \qquad y = x^3$$

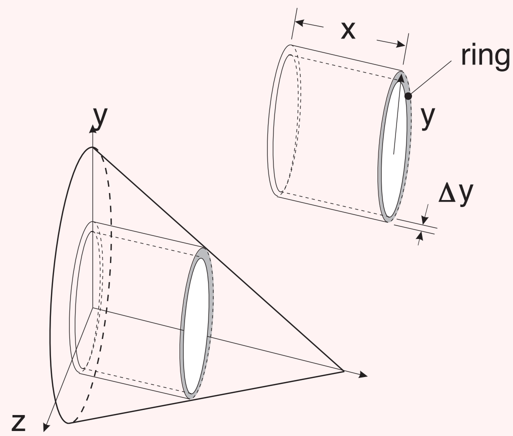
Example: 3.2

Find the volume of a cone with base radius $\boldsymbol{R}$ and height $\boldsymbol{h}$, rotated about the $\boldsymbol{x}$ axis using the disk method.



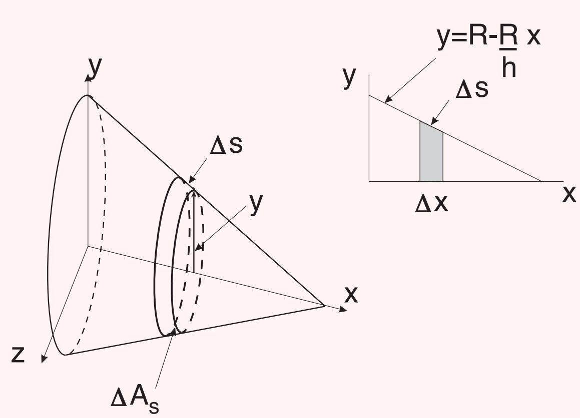
Example: 3.3

Find the volume of a cone with base radius $\boldsymbol{R}$ and height $\boldsymbol{h}$, rotated about the $\boldsymbol{x}$ axis using the shell method.

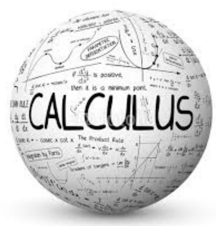


Example: 3.4

Find the surface area of a cone with base radius $\boldsymbol{R}$ and height $\boldsymbol{h}$, rotated about the $\boldsymbol{x}$ axis.



Double Integrals



Reading

Trim 13.1 $\longrightarrow$ *Double Integrals and Double Iterated Integrals*

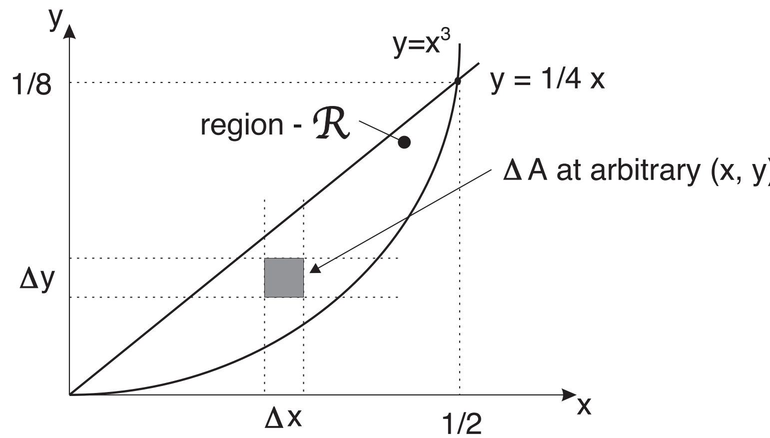
13.2 $\longrightarrow$ *Eval. of Double Integrals by Double Iterated Integrals*

13.7 $\longrightarrow$ *Double Iterated Integrals in Polar Coordinates*

Assignment

web page $\longrightarrow$ *assignment #7*

Cartesian Coordinates



Find the area in the +'ve quadrant bounded by $y = \frac{1}{4}x$ and $y = x^3$.

The basic area element in 2D is

$$\Delta A = \Delta x \cdot \Delta y$$

We can build this area up into a strip by summing over Δy, keeping x fixed.

$$\Delta A_{strip} = \left(\sum_{y=x^3}^{1/4x} \Delta y \right)_{fixed\ x} \Delta x$$

Sum up all Δx strips to get the total area

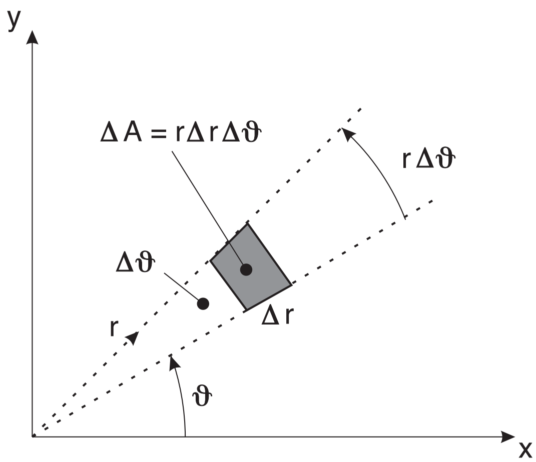
$$A = \sum_{x=0}^{1/2} \left[\sum_{y=x^3}^{1/4x} \Delta y \right] \Delta x$$

In the limit as $\Delta x \to dx$ and $\Delta y \to dy$ we get a double integral as follows

$$A = \int_{x=0}^{1/2} \left(\int_{y=x^3}^{1/4x} dy \right) dx$$

Polar Coordinates

In Cartesian coordinates our area element was $\Delta A = \Delta x \Delta y$, which in differential form gave us



$$A = \int\int_{\mathcal{R}} dxdy$$

We can change the principal coordinates into polar coordinates by transforming x and y into r and θ.

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

The Polar coordinate area element becomes

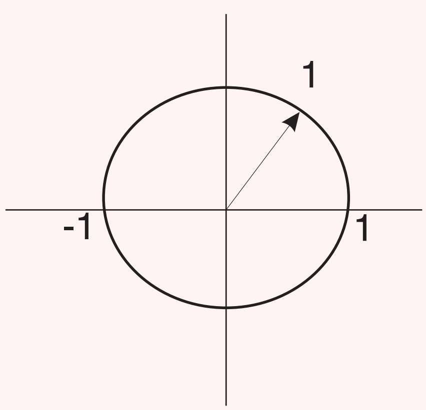
$$\Delta A = r\Delta r\Delta\theta$$

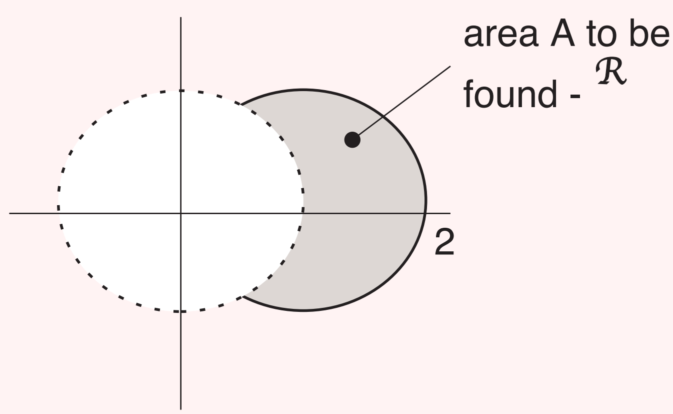
when integrated becomes

$$A = \int\int_{\mathcal{R}} rdrd\theta$$

Example: 3.6

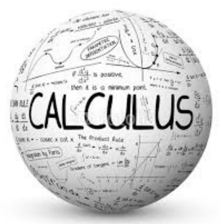
Find the area in the +'ve quadrant bounded by 2 circles





$$\begin{aligned} x^2 + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

Surface Areas from Double Integrals



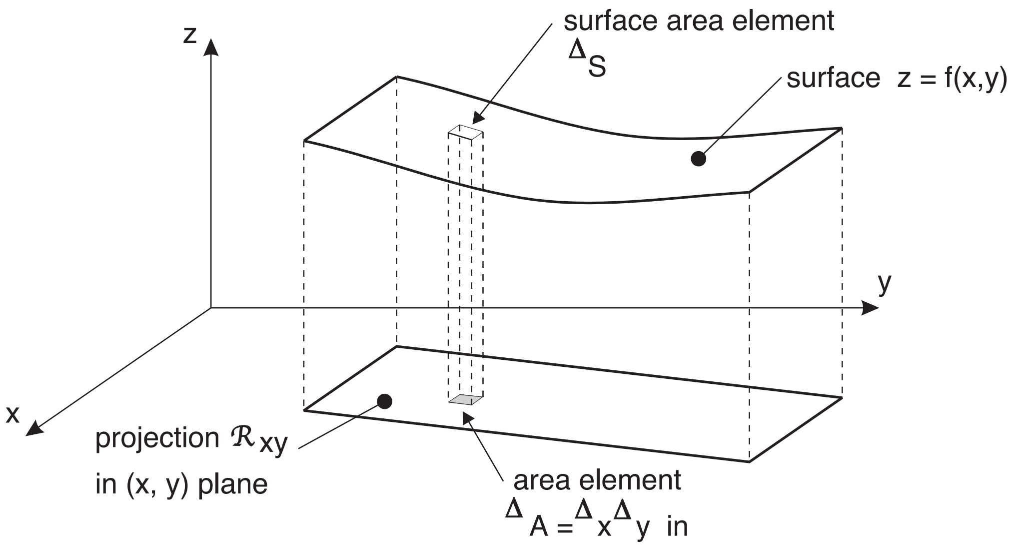
Reading

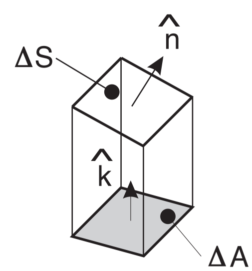
Trim 13.3 ⟶ *Areas and Volumes of Solids of Revolution*

13.6 ⟶ *Surface Area*

Assignment

web page ⟶ *assignment #7*





How is ΔS related to ΔA? Imagine shining a light vertically down through ΔS to get ΔA.

1. the surface is defined as $z = f(x, y)$

2. redefine as $F = z - f(x, y)$ where the surface is given as $F = 0$
   - $F > 0$ and $F < 0$ will the regions above and below the surface, respectively

3. the gradient of the function F is given as

$$\nabla F = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)$$

   ∇F is the perpendicular to the surface and the perpendicular to the tangent planes

$$\vec{n} = \nabla F$$

4. get the unit normal vector as follows

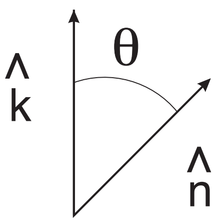
$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{-\hat{i}\dfrac{\partial f}{\partial x} - \hat{j}\dfrac{\partial f}{\partial y} + \hat{k}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + 1}}$$

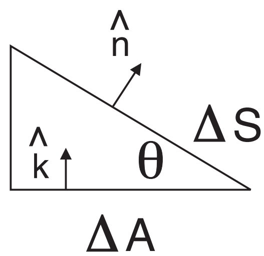
5. find the component of the ΔS surface projected onto $\hat{k}$

   from Trim 12.5 we know that

   $$\Delta A = \cos\theta \Delta S$$

   Note, when $\theta = 0 \quad \Rightarrow \quad \Delta A = \Delta S$ (this is the surface parallel to the xy plane.





In general,

$$\Delta A = \underbrace{\cos\theta}_{\hat{n}\cdot\hat{k}} \Delta S$$

$$\hat{n} \cdot \hat{k} = |\hat{n}||\hat{k}| \cos\theta = \cos\theta$$

$$\Delta A = \Delta S(\hat{n} \cdot \hat{k}) = \Delta S \frac{1}{|\nabla F|}$$

since $\hat{n} \cdot \hat{k}$ produces a numerator of $\hat{k} \cdot \hat{k} = 1$ and a denominator of $|\nabla F|$

Rearranging the above equation, we can solve for ΔS. In the limit

$$dS = \underbrace{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}_{|\nabla F|} \underbrace{dxdy}_{dA}$$

Given the surface $z = f(x, y)$, the surface area is

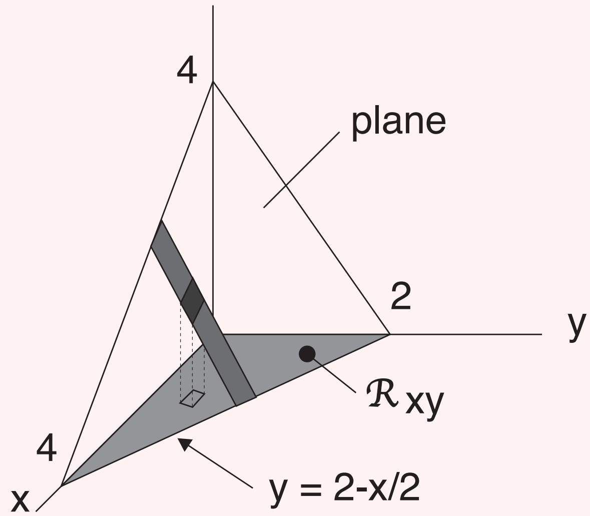
$$\boxed{S = \int\int_{\mathcal{R}_{xy}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy}$$

where $\mathcal{R}_{xy}$ is the projection of the $f(x, y)$ surface down onto the (x, y) plane.

While this is the most common form of the equation, we could also find S by projecting onto another coordinate plane. Sometimes it is more convenient to do it this way. See Trim 14.6 for applicable equations.

Example: 3.7

Find the surface area in the +'ve octant for $z = f(x, y) = 4 - x - 2y$.



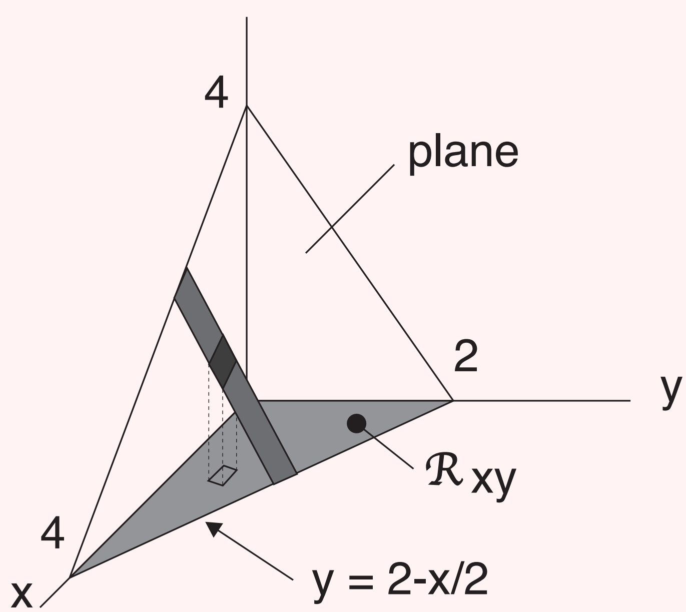
Example: 3.8

Given the sphere, $x^2 + y^2 + z^2 = a^2$, derive the formula for surface area.

Example: 3.9

Find the volume formed in the +'ve octant between the coordinate planes and the surface

$$z = f(x, y) = 4 - x - 2y$$

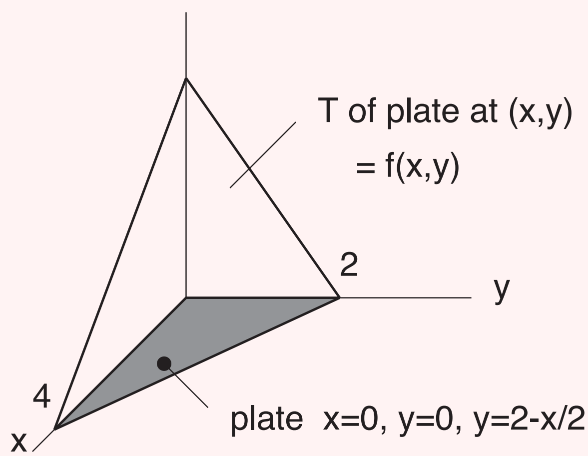


Example: 3.10a

Find the mean value of $y = f(x) = \sin x$ in the domain $x = 0$ to $x = \pi$.

Example: 3.10b

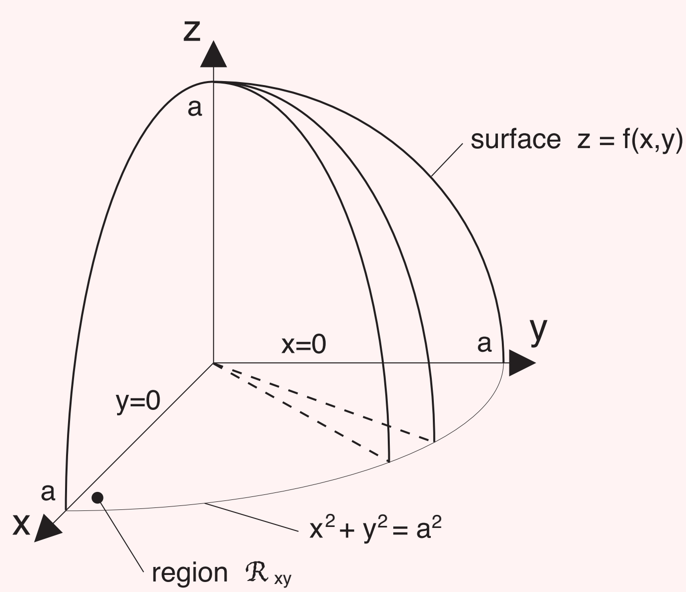
Find the mean value of temperature for $T = f(x, y) = 4 - x - 2y$.



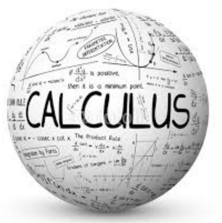
Example: 3.10c

Derive the formula for the volume of revolution. for the following sphere:

$x^2 + y^2 + z^2 = a^2$.



Triple Integrals



Reading

Trim 13.8 $\longrightarrow$ *Triple Integrals and Triple Iterated Integrals*

13.9 $\longrightarrow$ *Volumes*

Assignment

web page $\longrightarrow$ *assignment #8*

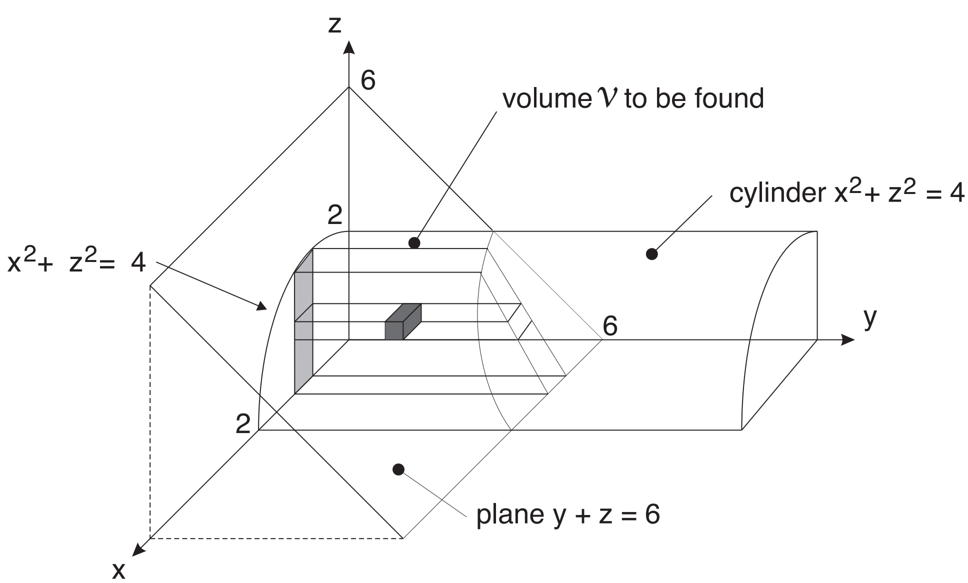
Volume Calculations in Cartesian Coordinates

The triple integral can be identified as

$$\int\int\int_{\mathcal{V}} \underbrace{dx\;dy\;dz}_{d\mathcal{V}\,-\,volume\ element} \qquad \text{or} \qquad \int\int\int_{\mathcal{V}} f\;dx\;dy\;dz$$

add up the $d\mathcal{V}$ elements in x, y, z directions, i.e. a triple sum.

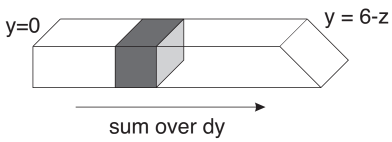
Consider the solid defined by $x^2 + z^2 = 4$ in the positive octant. Find the volume of this solid between the coordinate planes and the plane $y + z = 6$.



Start with a volume element at arbitrary (x, y, z) in space inside $d\mathcal{V}$

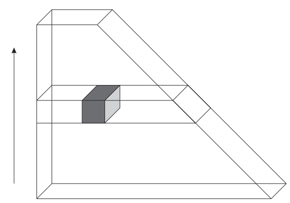
$$d\mathcal{V} = dx\ dy\ dz$$

Build up a column - sum over y keeping x, z constant.



$$\text{column volume} = \left(\int_{y=0}^{6-z} dy\right) dxdz$$

Build up a slice - sum columns over z, keeping y, x fixed.



$$\text{slice volume} = \left[\int_{z=0}^{\sqrt{4-x^2}} \left(\int_{y=0}^{6-z} dy\right) dz\right] dx$$

Finally sum the slices over x

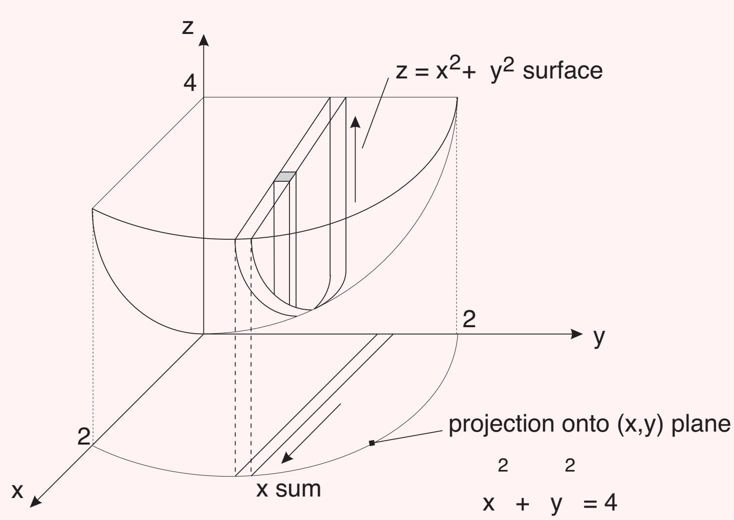
$$\mathcal{V} = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{6-z} dydzdx$$

Evaluation of the integral gives

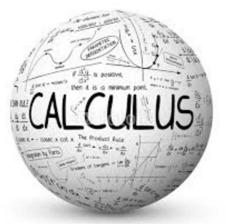
$$\begin{aligned}\mathcal{V} &= \int_0^2 \int_0^{\sqrt{4-x^2}} (6-z)dzdx = \int_0^2 6\sqrt{4-x^2} - \frac{(\sqrt{4-x^2})^2}{2} dx \\ &= 6\int_0^2 \sqrt{4-x^2}dx - \frac{1}{2}\int_0^2 (4-x^2)dx = 6\pi - \frac{8}{3} \approx 16.18 \qquad \text{use tables if necessary}\end{aligned}$$

Example: 3.11

Find the volume of the paraboloid, $z = x^2 + y^2$ for $0 \leq z \leq 4$. Consider only the +'ve octant, i.e. $1/4$ of the volume.



Volume Calculations in Cylindrical and Spherical Coordinates



Reading

Trim 13.11 $\longrightarrow$ *Triple Iterated Integrals in Cylindrical Coordinates*

13.12 $\longrightarrow$ *Triple Iterated Integrals in Spherical Coordinates*

Assignment

web page $\longrightarrow$ *assignment #8*

Cylindrical Coordinates

point: $\boldsymbol{P(r,\theta,z)}$ i.e. polar in $\boldsymbol{x,y}$ plane plus $\boldsymbol{z}$

volume element: $\boldsymbol{d\mathcal{V} = r\ dr\ d\theta\ dz}$

based on links to Cartesian coordinates

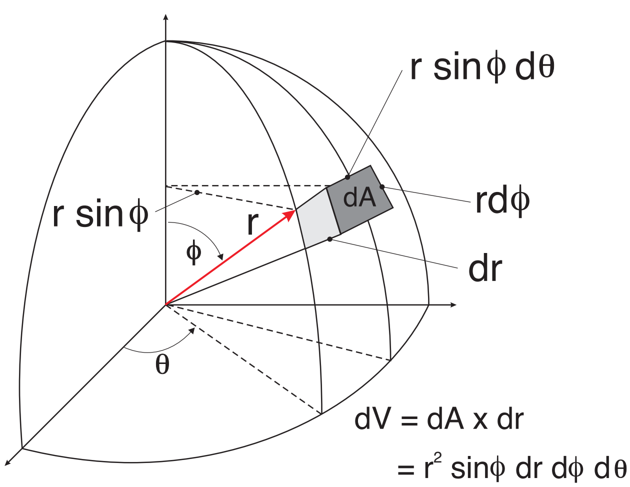
$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \\ z &= z \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

where $\boldsymbol{0 \le r, z \le \infty}$ and $\boldsymbol{0 \le \theta \le 2\pi}$

Typically we build up column, wedge slice and then the total volume, given as $\int\int\int \boldsymbol{r dr d\theta dz}$

The math operations are easier when we have axi-symmetric systems, i.e. cylinders and cones

Spherical Coordinates



point: $\boldsymbol{P(r,\theta,\phi)}$

volume element: $\boldsymbol{d\mathcal{V} = \underbrace{(r\sin\phi d\phi)}_{height}\underbrace{rdrd\theta}_{area}}$

based on links to Cartesian coordinates

$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \\
\theta &= \tan^{-1}(y/x) \\
\phi &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
x &= r\sin\phi\cos\theta \\
y &= r\sin\phi\sin\theta \\
z &= r\cos\phi
\end{aligned}
$$

where $0 \le r \le \infty$; $0 \le \theta \le 2\pi$; $0 \le \phi \le \pi$. Note: for $0 \le \phi \le \pi$ the $sin\phi$ is always +'ve for $d\mathcal{V}$ +'ve.

The solution procedure involves building up columns, slices as before to obtain the total volume, given as

$$\boxed{\int\int\int r^2 \ \sin\ \phi \, dr \, d\theta \, d\phi}$$

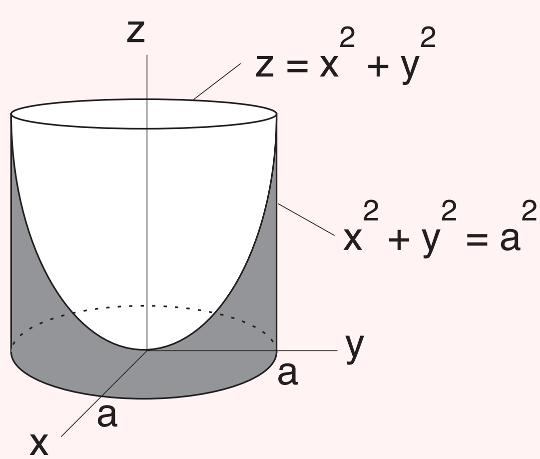
Example: 3.12

Find the volume bounded by a cylinder,

$$x^2 + y^2 = a^2$$

and a paraboloid,

$$z = x^2 + y^2$$



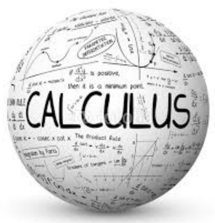
Spherical Coordinate Example

Example: 3.13

Derive a formula for the volume of a sphere with radius, a

$$x^2 + y^2 + z^2 = a^2$$

Moments of Area/ Mass / Volume



Reading

Trim 13.5 $\longrightarrow$ *Centres of Mass and Moments of Inertia*

13.10 $\longrightarrow$ *Centres of Mass and Moments of Inertia*

Assignment

web page $\longrightarrow$ *assignment #9*

Centroids, Centers of Mass etc.

2-D case: thin plate of constant thickness

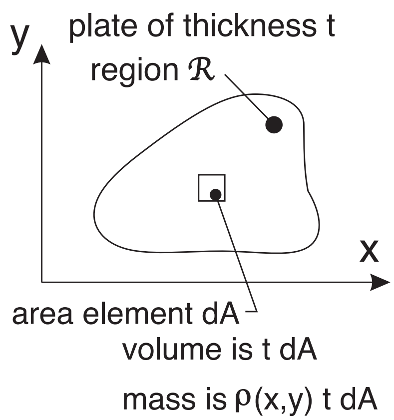
Sometimes, single integrals work, as in a 2-D case, where the thickness is given as t and is constant or a function of position as $t(x, y)$. The material density is given as ρ (kg/m^3), again constant or a function of position as $\rho(x, y)$. We sometimes use the mass per unit area of the plate, $\rho^* = \rho \cdot t$ (kg/m^2).

	area	**mass**
basic element	$dA = dx\,dy$	$dM = \rho\,t\,dx\,dy$ or $\rho^*\,dx\,dy$
total area	$A = \int\int_{\mathcal{R}} dx\,dy$	$M = \int\int_{\mathcal{R}} dM = \int\int_{\mathcal{R}} \rho\,t\,dx\,dy$
	first moment of area	**first moment of mass**
	(*weight by distance from axis*)	
about $y-$ axis	$x\,dA = x\,dx\,dy$	$x\,dM = x\rho\,t\,dx\,dy$
total	$F_y = \int\int_{\mathcal{R}} x\,dA$	$\int\int_{\mathcal{R}} x\,dM$
about $x-$ axis	$F_x = \int\int_{\mathcal{R}} y\,dA$	$\int\int_{\mathcal{R}} y\,dM$
	centroid coordinates	**center of mass coordinates**
	$\overline{x} = \dfrac{\int\int_{\mathcal{R}} x\,dA}{A}$	$\overline{x_c} = \dfrac{\int\int_{\mathcal{R}} x\,dM}{M}$
	$\overline{y} = \dfrac{\int\int_{\mathcal{R}} y\,dA}{A}$	$\overline{y_c} = \dfrac{\int\int_{\mathcal{R}} y dM}{M}$
second moments		
	$\int\int_{\mathcal{R}} x^2\,dA$	$\int\int_{\mathcal{R}} x^2\,dM$
	$\int\int_{\mathcal{R}} y^2\,dA$	$\int\int_{\mathcal{R}} y^2\,dM$

3-D case:

We use the same basic ideas but the basic element is now $\mathcal{V} = dxdydz$

2-D Objects



Quantities of interest in applications such as dynamics.
Area: $A = \int\int_{\mathcal{R}} dA \ (Volume = tA)$

Mass: $M = \int\int_{\mathcal{R}} \rho(x,y)tdA$
where $\rho(x,y)$ = **density of material** in (kg/m^3) at point (x,y)

Centroid = "geometrical center" of object

$$\overline{x} = \frac{\int\int_{\mathcal{R}} xdA}{A}$$ **1st moment of area** about $y-$ axis

$$\overline{y} = \frac{\int\int_{\mathcal{R}} ydA}{A}$$ **1st moment of area** about $x-$ axis

---

Center of Mass: useful in dynamics problems

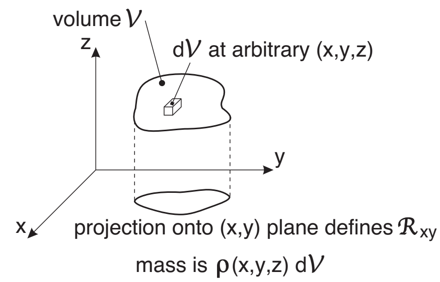
$$\overline{x_c} = \frac{\int\int_{\mathcal{R}} xdm}{M} = \frac{\int\int_{\mathcal{R}} x\rho(x,y)tdA}{M}$$

$$\overline{y_c} = \frac{\int\int_{\mathcal{R}} ydm}{M} = \frac{\int\int_{\mathcal{R}} y\rho(x,y)tdA}{M}$$

Note: that if the object density is uniform, then the centroid and center of mass are the same.

2nd Moments of Area and Mass:
⟶ Moments of Inertia
2nd moment of area about: $y - axis$
$I_y = \int\int_{\mathcal{R}} x^2 dA$
2nd moment of mass about: $y - axis$
$I_y = \int\int_{\mathcal{R}} x^2 \rho(x,y)tdA$
(similar formulas for I_x about the $x - axis$)

3-D Objects



Quantities of interest in applications such as dynamics.
Volume: $\mathcal{V} = \int\int\int_{\mathcal{V}} d\mathcal{V}$

Mass: $M = \int\int\int_{\mathcal{V}} \rho(x,y,z)d\mathcal{V}$
where $\rho(x,y,z)$ = **density of material** in (kg/m^3) at point (x,y,z)

Centroid = "geometrical center" of object

$$\overline{x} = \frac{\int\int\int_{\mathcal{V}} xd\mathcal{V}}{\mathcal{V}}$$ **1st moment of volume** about $y - z$ plane

$$\overline{y} = \frac{\int\int\int_{\mathcal{V}} yd\mathcal{V}}{\mathcal{V}}$$ **1st moment of volume** about $x - z$ plane

$$\overline{z} = \frac{\int\int\int_{\mathcal{V}} zd\mathcal{V}}{\mathcal{V}}$$ **1st moment of volume** about $x - y$ plane

---

Center of Mass: useful in dynamics problems

$$\overline{x_c} = \frac{\int\int\int_{\mathcal{V}} x\rho(x,y,z)d\mathcal{V}}{M}$$

similar formulas for $\overline{y}_c$ and $\overline{z}_c$

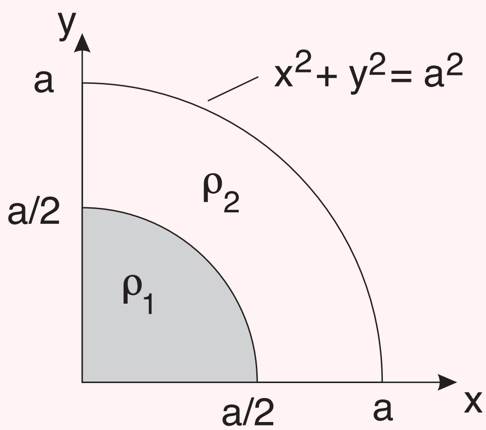
2nd Moments of Area and Mass:
⟶ Polar Moments of Inertia
volume moment about: $y - axis$
$J_y = \int\int\int_{\mathcal{V}} (x^2 + z^2)d\mathcal{V}$
mass moment about: $y - axis$
$J_y = \int\int\int_{\mathcal{V}} (x^2 + z^2)\rho(x,y,z)d\mathcal{V}$
(similar formulas for J_x about the $x - axis$)

and

(similar formulas for J_z about the $z - axis$)

Example: 3.14

Find the centroid, center of mass and the 1st moment of mass for a quarter circle of radius a with an inner circle of radius $a/2$ made of lead with a density of $\rho_1 = 11,000\ kg/m^3$ and an outer circle of radius a made aluminum with a density of $\rho_2 = 2,500\ kg/m^3$. The thickness is uniform throughout at $t = 10\ mm$.



$$\begin{aligned}\rho_1^* &= 11\ g/cm^2 = 110\ kg/m^2\\ \rho_2^* &= 2.5\ g/cm^2 = 25\ kg/m^2\end{aligned}$$

Example: 3.15

Find the area of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$

Example: 3.16

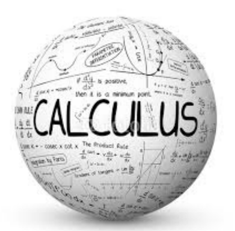
Find the moment of inertia about the $y-$ axis of the area enclosed by the cardioid $r = a(1 - \cos\theta)$

Example: 3.17

Find the center of gravity of a homogeneous solid hemisphere of radius a

Vector Calculus

Vector Fields



Reading
Trim 14.1 $\longrightarrow$ *Vector Fields*

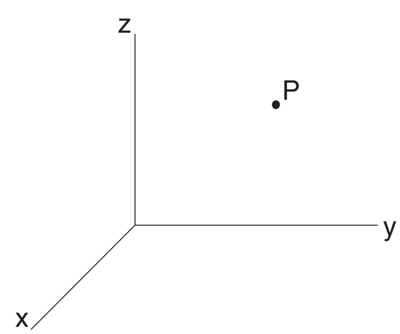
Assignment
web page $\longrightarrow$ *assignment #9*

Chapter 14 will examine a vector field.

For example, if we examine the temperature conditions in a room, for every point P in the room, we can assign an air temperature, T, where

$$T = f(x, y, z)$$

This is a scalar function or scalar field.

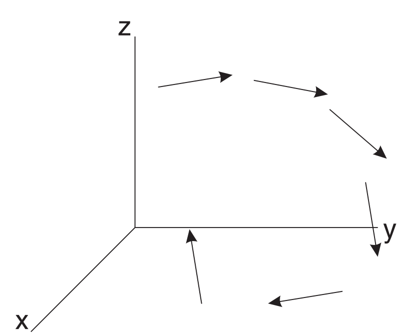


However, suppose air is moving around in the room and at every point P, we can assign an air velocity vector, $\vec{V}(x, y, z)$, where

$$\vec{V}(x, y, z) = \hat{i}\, u(x, y, z) + \hat{j}\, v(x, y, z) + \hat{k}\, w(x, y, z)$$

where the right side of the above equation consists of x, y, z components at a point, with each component being a function of (x, y, z). To describe $\vec{V}$ in the room, we need to keep track of the 3 scalar function, u, v, w at each $f(x, y, z)$.

$\vec{V}(x, y, z)$ is a vector function or a vector field.



Gradient, Divergence and Curl Operations

The basic operator is the del operator given as

2D

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$$

3D

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The del operator operates on a scalar function, such as $T = f(x, y, z)$ or on a vector function, such as $\vec{F} = \hat{i}P + \hat{j}Q + \hat{k}R$ (force) or $\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$ (velocity).

We will examine three operations is more detail

1. **gradient:** ∇ operating on a scalar function

   Examples include the temperature in a 2D plate, $T = f(x, y)$

   $$\boxed{\nabla T = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}}$$

   or for a 3D volume, such as a room

   $$\boxed{\nabla T = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}}$$

   The physical meaning was given in Chapter 13. ∇T is a vector perpendicular to the T contours that points "uphill" on the contour plot or level surface plot.

2. **divergence:** ∇ dotted with a vector function $\longrightarrow \quad \nabla \cdot \vec{V}$

   We can define the velocity in a 3D room as

   $$\vec{V}(x, y, z) = \hat{i}u(x, y, z) + \hat{j}v(x, y, z) + \hat{k}u(x, y, z)$$

   $$\text{DIVERGENCE of } \vec{V} = \text{div } \vec{V} = \nabla \cdot \vec{V}$$

   $$\boxed{\nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}u + \hat{j}v + \hat{k}w\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

   The DIVERGENCE of a vector is a scalar.

3. **curl:** the vector product of the del operator and a vector, $\boldsymbol{\nabla} \times \vec{\boldsymbol{V}}$, produces a vector

$$\text{curl of } \vec{V} = \text{curl } \vec{V} = \nabla \times \vec{V}$$

$$\begin{aligned}
\nabla \times \vec{V} &= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \times \left(\hat{i}u + \hat{j}v + \hat{k}w\right) \\
&\equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\
&= \hat{i}\left(\underbrace{\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}}_{V_1}\right) + \hat{j}\left(\underbrace{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}}_{V_2}\right) + \hat{k}\left(\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{V_3}\right) \\
&= \hat{i}\,V_1 + \hat{j}\,V_2 + \hat{k}\,V_3
\end{aligned}$$

The **3D** case gives

$$\boxed{\text{curl } \vec{V} = \hat{i}\left(\underbrace{\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}}_{V_1}\right) + \hat{j}\left(\underbrace{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}}_{V_2}\right) + \hat{k}\left(\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{V_3}\right)}$$

The **2D** case gives

$$\boxed{\text{curl } \vec{V} = \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}$$

4. **Laplacian:** $(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})$ operation.

   A primary example of the Laplacian operator is in determining the conduction of heat in a solid. Given a 3D temperature field $T(x, y, z)$, the Laplacian is

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} = 0}$$

2D example

Consider a 2D heat flow field, $\vec{q}(x, y)$.

Look at a small differential element, $\Delta x \Delta y$. In steady state, heat flows in is equivalent to heat flow out. Therefore

$$\nabla \cdot \vec{q} = 0 \qquad (1)$$

But $\vec{q}$ is related to temperature by Fourier's law

$$\vec{q} = -k\nabla T \qquad (2)$$

Combining (1) and (2)

$$\nabla \cdot (-k\nabla T) = 0 \rightarrow \nabla^2 T = 0$$

This is Laplace's equation in 2D, which gives the steady state temperature field.

Example 4.1

Show for the 3D case $f(x, y, z)$ that **curl grad** $f = 0$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

holds for any function $f(x, y, z)$, for instance

$$f(x, y, z) = x^2 + y^2 + y\sin x + z^2$$

Conservative Force Fields and the $\operatorname{curl} \operatorname{grad} f = 0$ identity

Suppose we have a conservative field $\vec{F}(x, y, z)$. We know $\vec{F}$ is irrotational, i.e.

$$\nabla \times \vec{F} = 0 \qquad (1)$$

(zero work in a closed path in the field)

But the identity says

$$\nabla \times (\nabla \phi) = 0 \qquad (2)$$

always holds when $\phi(x, y, z)$ is a scalar function.

Comparing (1) and (2) $\rightarrow$ for a Conservative Force Field, we can always find a scalar function $\phi(x, y, z)$ such that

$$\vec{F} = \nabla \phi$$

where ϕ is called the scalar potential function.

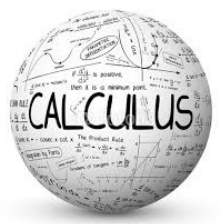
Sometimes, for convenience, we introduce a negative sign

$$\vec{F} = -\nabla u$$

The following statements are equivalent

$\vec{F}$ is a conservative force field	$\Longleftrightarrow$	net work when a particle moves through $\vec{F}$ around a closed path in space is zero
	$\Longleftrightarrow$	$\nabla \times \vec{F} = 0$ irrotational
	$\Longleftrightarrow$	a scalar function $\phi(x, y, z)$ can be found such that $\vec{F} = \nabla \phi$ or a function $u(x, y, z)$ can be found such that $\vec{F} = -\nabla u$

Line Integrals of Scalar Functions

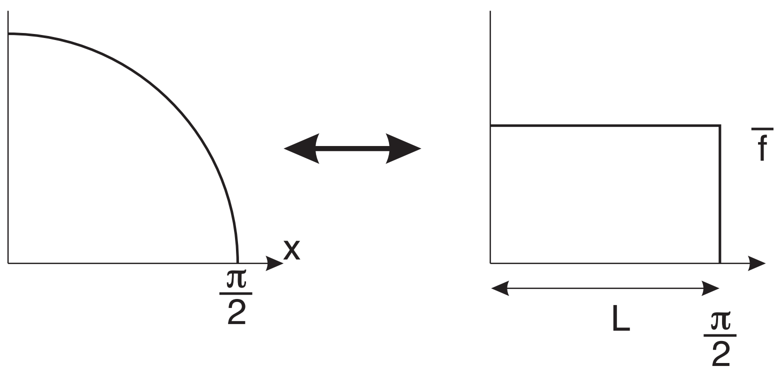


Reading
Trim 14.2 $\longrightarrow$ *Line Integrals*

Assignment
web page $\longrightarrow$ *assignment #10*

One place that line integrals often come up is in the computation of averages of a function.

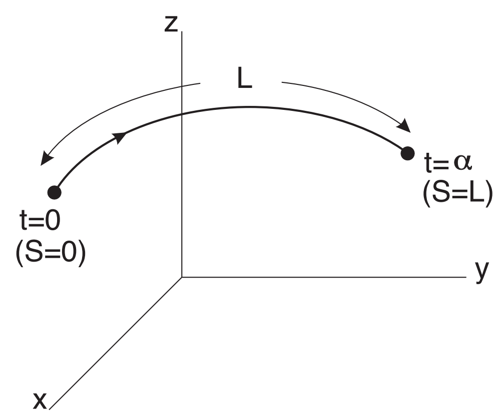
2D Case



$$\boxed{\overline{f} = \frac{1}{L}\int_{x=0}^{L} f(x)dx}$$

3D Case

The temperature in a room is given by $T = f(x, y, z)$. A curve C in the room is given by $x(t), y(t)$ and $z(t)$. If we measure temperature along curve C, what is the average temperature $\overline{T}$?



$$\overline{T} = \frac{1}{L}\underbrace{\int_C \overbrace{f(x,y,z)}^{value\ of\ f\ along\ C} \overbrace{dS}^{arc\ length\ of\ C}}_{line\ integral\ along\ C}$$

Calculation of the line integral along C

$$\int_{t=0}^{\alpha} \underbrace{f[x(t), y(t), z(t)]}_{f\ values\ along\ C} \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}_{dS\ along\ C}\ dt$$

Example 4.2

Given a 3D temperature field

$$T = f(x, y, z) = 8x + 6xy + 30z$$

find the average temperature, $\overline{T}$ along a line from $(0, 0, 0)$ to $(1, 1, 1)$.

1. if we have an explicit equation for a planar curve

   $$C: \qquad y = g(x)$$

   we can reduce

   $$\int_C f\,dS \qquad \text{to} \qquad \int \text{fnc of } x \;\; dx \quad \text{or} \quad \int \text{fnc of } y \;\; dy$$

   we do not have to always use the parametric equations.

2. value of $\int_C f\ dS$ depends on

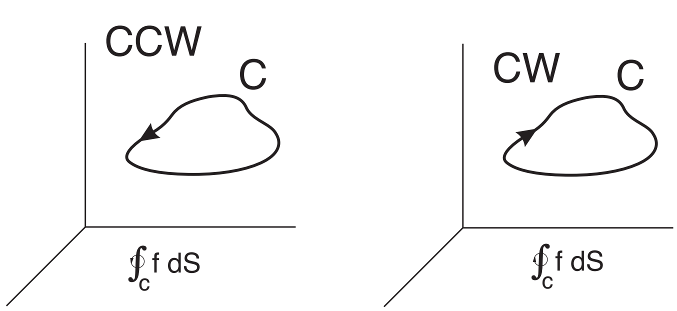
   (i) function f

   (ii) curve C in space

   (iii) direction of travel

   $$\int_A^B f\ dS = -\int_B^A f\ dS$$

3. notation - sometimes C is a closed loop in space



- evaluate once around the loop CCW or CW
- evaluation method is the same as the example

Example: 4.3a

Suppose the temperature near the floor of a room (say at $z = 1$) is described by

$$T = f(x, y) = 20 - \frac{x^2 + y^2}{3} \qquad \text{where} \quad -5 \le x \le 5$$

$$-4 \le y \le 4$$

What is the average temperature along the straight line path from $A(0, 0)$ to $B(4, 3)$.

Example: 4.3b

What is the average room temperature along the walls of the room?

$$\overline{T} = \frac{\oint_C f(x, y) dS}{\oint_C dS}$$

where the closed curve C is defined in 4 sections

$C_1 \quad y = -4 \quad x = t \qquad -5 \le t \le 5$

$C_2 \quad x = 5 \quad y = t \qquad -4 \le t \le 4$

$C_3 \quad y = 4 \quad x = 5 - t \quad 0 \le t \le 10$

$C_4 \quad x = -5 \quad y = 8 - t \quad 0 \le t \le 8$

Example: 4.3c

What is the average temperature around a closed circular path $\rightarrow \quad x^2 + y^2 = 9$?

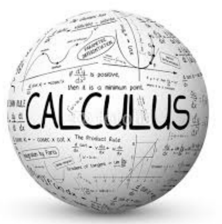
where C is a closed circular path

$$\overline{T} = \frac{\oint_C f(x, y) dS}{\oint_C dS} \qquad \text{where} \quad x(t) = 3 \cos t$$

$$y(t) = 3 \sin t$$

for $0 \le t \le 2\pi$.

Line Integrals of Vector Functions



Reading

Trim 14.3 $\longrightarrow$ *Line Integrals Involving Vector Functions*

14.4 $\longrightarrow$ *Independence of Path*

Assignment

web page $\longrightarrow$ *assignment #10*

Let's examine work or energy in a force field.

$$W = \vec{F} \cdot \vec{d}$$

Now consider a particle moving along a curve C is a 3D force field.

$$W = \int_C \underbrace{\vec{F}(x, y, z)}_{force\ value\ evaluated\ along\ C} \cdot \underbrace{d\vec{r}}_{displacement\ along\ C} \qquad (1)$$

This is a line integral of the vector $\vec{F}$ along the curve C in 3D space.

In component form:

$$\begin{aligned} \vec{F} &= \hat{\imath}P + \hat{\jmath}Q + \hat{k}R \\ d\vec{r} &= \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz \end{aligned}$$

$$W = \int_C P dx + Q dy + R dz \qquad (2)$$

Equations (1) and (2) are equivalent.

Use the equation of curve C (either in an explicit form, i.e. $y = f(x)$ etc., or in a parametric form) to reduce (2) to $\int_{t=0}^{\alpha} g(t)dt$ or $\int_{x=n}^{\beta} h(x)dx$ etc.

Example 4.4

Given a force field in 3D:

$$\vec{F} = \hat{\imath}(3x^2 - 6yx) + \hat{\jmath}(2y + 3xz) + \hat{k}(1 - 4xyz^2)$$

What is the work done by $\vec{F}$ on a particle (i.e. energy added to the particle) if it moves in a straight line from $(0, 0, 0)$ to $(1, 1, 1)$ through the force field.

Notes

1. If we have an explicit equation for the curve, we can sometimes reduce to a form $\int (\textbf{fnc of } \boldsymbol{x})\, \boldsymbol{dx}$ etc. There is no need for parametric equations.

2. The work term $\boldsymbol{W}$ can be either +'ve or -'ve.
   In a +'ve form, $\int \vec{\boldsymbol{F}}$ and $\int \boldsymbol{d\vec{r}}$, are in the same direction, where the work done by the force energy is added to the object by $\vec{\boldsymbol{F}}$.
   In the -'ve form, $\vec{\boldsymbol{F}}$ opposes the displacement. The energy is removed from the object.
   +'ve $\boldsymbol{W} = \int_C \vec{\boldsymbol{F}} \cdot \boldsymbol{d\vec{r}}$

3. closed path notation

$$\boldsymbol{W} = \oint_{CCW} \vec{\boldsymbol{F}} \cdot \boldsymbol{d\vec{r}} \quad \textbf{once CCW around loop}$$
$$\boldsymbol{W} = \oint_{CW} \vec{\boldsymbol{F}} \cdot \boldsymbol{d\vec{r}} \quad \textbf{once CW around loop}$$

4. A special case is the conservative force field

$$\oint \vec{\boldsymbol{F}} \cdot \boldsymbol{d\vec{r}} = \boldsymbol{0}$$

   We know that $\boldsymbol{\nabla} \times \vec{\boldsymbol{F}} = \boldsymbol{0}$. There is no work in a closed loop.
   We also know that $\boldsymbol{\nabla}\boldsymbol{\phi} = \boldsymbol{F}$, which is integrated gives

$$\int_C \vec{\boldsymbol{F}} \cdot \boldsymbol{d\vec{r}} = \boldsymbol{\phi_1} - \boldsymbol{\phi_2}$$

   Therefore for a conservative force field, the work is a function of the end points not the path.
   This is the same for <u>any</u> $\boldsymbol{C}$ connecting the same 2 end points.
   $\boldsymbol{W}$ is always the same number if $\vec{\boldsymbol{F}}$ is conservative.

Example: 4.5a

The gravitational force on a mass, m, due to mass, M, at the origin is

$$\vec{F} = -G\frac{Mm\vec{r}}{|\vec{r}|^3} = -K\frac{\vec{r}}{|\vec{r}|^3} \qquad \text{where} \quad K = GMm$$

The vector field is given by:

$$\vec{F}(x,y,z) = \hat{i}P + \hat{j}Q + \hat{k}R \qquad \text{where} \quad P = -\frac{Kx}{(x^2+y^2+z^2)^{3/2}}$$

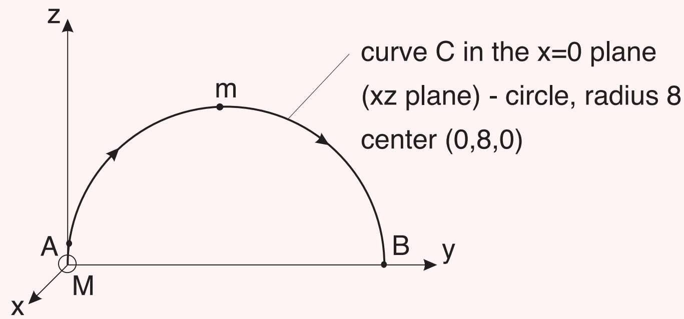
$$Q = -\frac{Ky}{(x^2+y^2+z^2)^{3/2}}$$

$$R = -\frac{Kz}{(x^2+y^2+z^2)^{3/2}}$$

Compute the work, W, if the mass, m moves from A to B along a semi-circular path in the (y,z) plane:

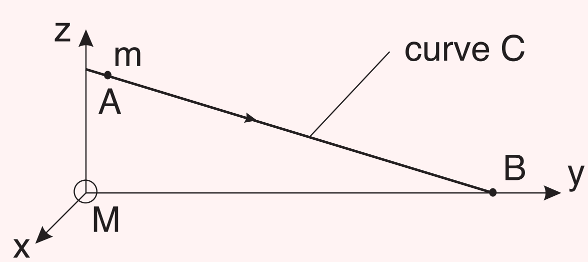
$$y^2 + z^2 = 16y \qquad \text{or} \qquad z = \sqrt{16y - y^2} \quad \text{and } x = 0$$

From $A(0, 0.1, 1.261)$ to $B(0, 16, 0)$

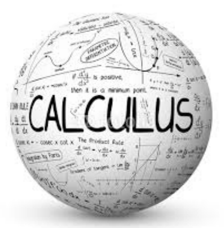


Example: 4.5b

Find the work to move through the same field, but following a straight line path from $A(0, 0.1, 1.261)$ to $B(0, 16, 0)$.



Conservative Force Fields



Reading
Trim 14.5 $\longrightarrow$ *Energy and Conservative Force Fields*

Assignment
web page $\longrightarrow$ *assignment #10*

Given a flow field in 3D space

$$\vec{F} = \hat{\imath}\,\underbrace{(2xz^3 + 6y)}_{P} + \hat{\jmath}\,\underbrace{(6x - 2yz)}_{Q} + \hat{k}\,\underbrace{(3x^2z^2 - y^2)}_{R}$$

Part a: Is the force field, $\vec{F}$, conservative?

Check to see if $\nabla \times \vec{F} = 0$ (i.e. irrotational $\vec{F}$?)

Part b: Compute the work done on an object if it goes around a CCW circular path of radius 1 for center point $P(2, 0, 3)$ with form

Part c: Find the scalar potential function $\phi(x, y, z)$

$$\vec{F} = \nabla\phi$$

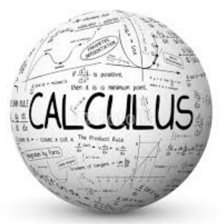
Part d: Use ϕ to verify part b).

$$W = \oint \vec{F} \cdot d\vec{r} = \oint \nabla\phi \cdot d\vec{r}$$

Part e:

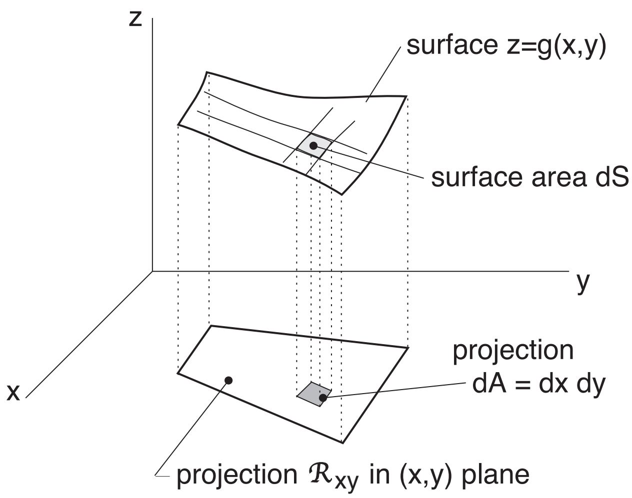
Find the work done if the object moves along C from $A(0, 0, 0)$ to $B(3, 4, -2)$ within $\vec{F}$.

Surface Integrals of Scalar Functions



Reading
Trim 14.7 $\longrightarrow$ *Surface Integrals*

Assignment
web page $\longrightarrow$ *assignment #11*



surface area of $z = g(x, y)$

$$S = \int\int_{\mathcal{R}_{xy}} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dxdy$$

Now suppose the surface $z = g(x, y)$ is within a 3D temperature field

$$T = f(x, y, z)$$

What is the average temperature measured over the surface $z = g(x, y)$?

Add up T for each dS area element and divide by the total area of S to get the average.

$$\overline{T} = \frac{\int\int_S f(x, y, z) dS}{Area\ of\ S}$$

The numerator is called the surface integral of $f(x, y, z)$ over the surface S (i.e. $z = g(x, y)$).

To evaluate

$$\int\int_S f dS = \int\int_{\mathcal{R}_{xy}} \underbrace{f[x, y, g(x, y)]}_{T\ values\ on\ the\ surface} \underbrace{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dxdy}_{dS\ area\ element}$$

This becomes

$$\int\int_{\mathcal{R}_{xy}} F(x, y) dxdy$$

Notes

1. sometimes it is easier if we switch to polar coordinates

$$\int\int_{\mathcal{R}_{xy}} F(x,y)dxdy \qquad \rightarrow \qquad \int\int_{\mathcal{R}_{xy}} H(r,\theta)rdrd\theta$$

   where $x = r\cos\theta$ and $y = r\sin\theta$.

2. notation = we have a closed surface in space, i.e. a sphere surface

$$g(x,y) \rightarrow z = \sqrt{a^2 - x^2 - y^2}$$

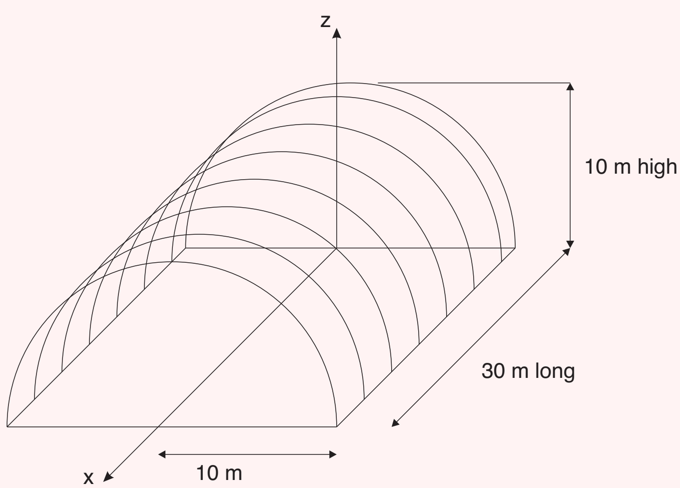
$$\oint\oint_S f(x,y,z)dS$$

Example: 4.6

Suppose the temperature variation (same for all (x,y)) in the atmosphere near the ground is

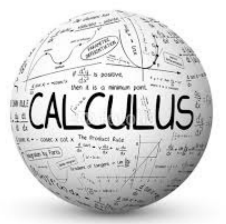
$$T(z) = 40 - \frac{z^2}{5}$$

where T is in $^\circ C$ and z is in m. Look at a cylindrical building roof as follows:



What is the air temperature in contact with the roof?

Surface Integrals of Vector Functions

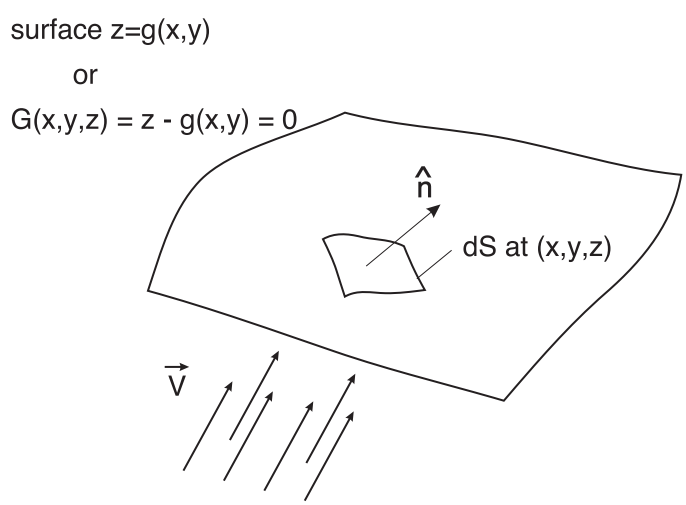


Reading
Trim 14.8 $\longrightarrow$ *Surface Integrals Involving Vector Fields*

Assignment
web page $\longrightarrow$ *assignment #11*

Given a full 3D velocity field, $\vec{V}(x, y, z)$ in space (i.e. air flow in a room).

Given some surface $z = g(x, y)$ within the flow $\rightarrow$ calculate the flow rate Q (m^3/s) crossing the surface S, we can write $G = z - g(x, y)$, where G is a constant since the surface is a level surface or a contour.



The basic idea is to consider the element dS of the surface at some arbitrary (x, y, z). Then compute the unit normal vector to dS

$$\hat{n} = (\pm)\frac{\nabla G}{|\nabla G|}$$

This will vary over S. The $(\pm)$ will be controlled by the direction of the flow.

The flow across dS is $\underbrace{(\vec{V} \cdot \hat{n})}_{component\ normal\ to\ surface} dS$.

Add up over all dS elements to get the total flow across the surface

$$Q = \int\int_S \vec{V} \cdot \hat{n}\, dS$$

This is called the surface integral of the vector field $\vec{V}$ over the surface S defined by $z = g(x, y)$.

The actual evaluation is similar to the last example.

- $\vec{V} \cdot \hat{n}$ will end up giving some integrand function f
- project dS onto (x, y) plane

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dxdy$$

$$\boxed{Q = \int\int_{\mathcal{R}_{x,y}} f(x, y)\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dxdy}$$

- then proceed as before.

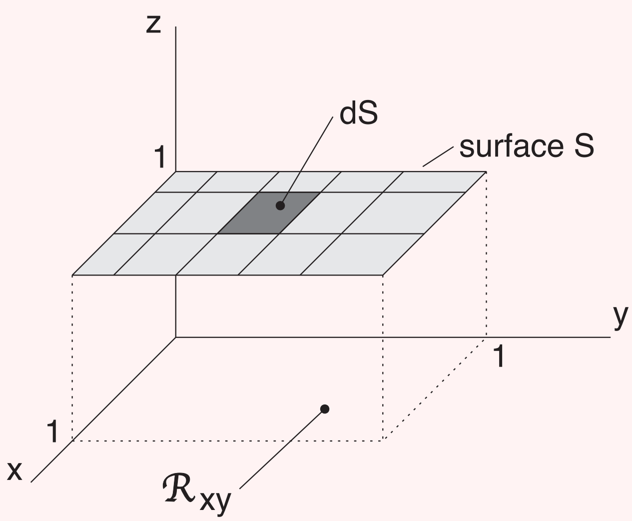
Example: 4.7

Given a velocity field in 3D space

$$\vec{V} = \hat{i}(2x + z) + \hat{j}(x^2y) + \hat{k}(xz) \qquad \text{find}$$

a) the flow rate Q (m^3/s) across the surface $z = 1$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$ in the +'ve z direction

b) the average velocity across the surface



Integral Theorems Involving Vector Functions

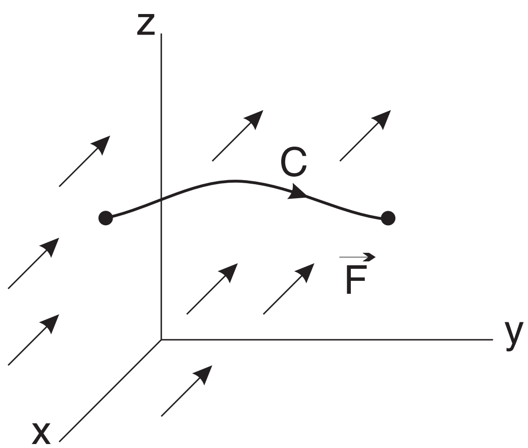
We will examine vector functions in 3D:

Force Field: $\vec{F}(x, y, z) = \hat{i}P + \hat{j}Q + \hat{k}R$

Vector Field: $\vec{V}(x, y, z) = \hat{i}u + \hat{j}v + \hat{k}w$

We have defined 2 types of integrals for such functions.

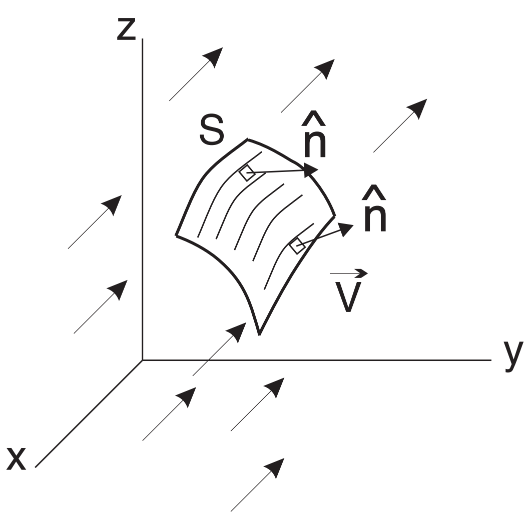
Line Integrals



$$\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz$$

This can be interpreted as work done by $\vec{F}$ field on an object that moves along C within the field.

Surface Integrals

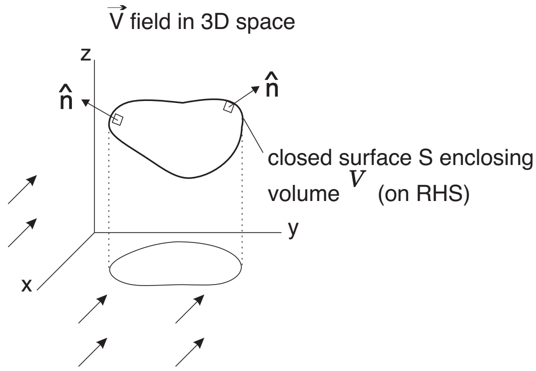


$$\int\int_S \vec{V} \cdot \hat{n}\, dS$$

This can be interpreted as the flow across surface S in the $\hat{n}$ direction due to the $\vec{V}$ field.

There are three theorems which state identities involving these types of integrals.

1. **Divergence Theorem**



Also called Green's theorem in space - this is the 2nd vector form of Green's theorem.

$$\boxed{\oint\oint_S \vec{V}\cdot\vec{n}\,dS = \int\int\int_{\mathcal{V}}(\nabla\cdot\vec{V})d\mathcal{V}}$$

where

$$\vec{V} = \text{velocity field}$$

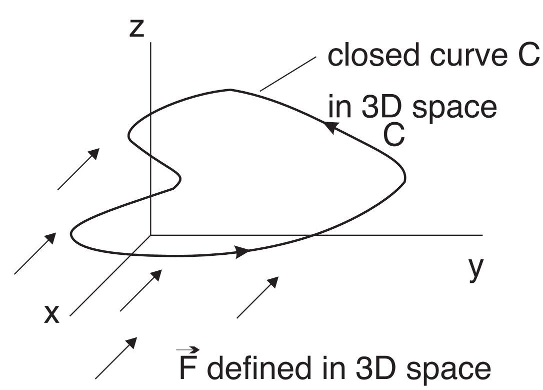
$$\mathcal{V} = \text{volume}$$

2. **Stokes Theorem**

Also called 1st vector from of Green's theorem.

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \int\int_S(\nabla\times\vec{F})\cdot\hat{n}\,dS}$$

where the surface S is any surface in 3D with C as a boundary.

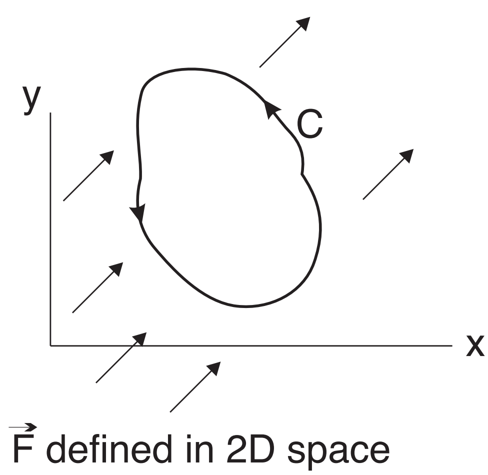


3. **Greens Theorem**

This is essentially a 2D statement of Stoke's theorem, where in 2D

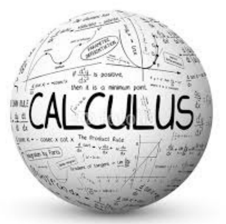
$$\vec{F} = \hat{i}P + \hat{j}Q$$

$$\nabla\times\vec{F} = \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$



$$\boxed{\oint_C Pdx + Qdy = \int\int_{\mathcal{R}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy}$$

Divergence Theorem



Reading
Trim 14.9 $\longrightarrow$ *The Divergence Theorem*

Assignment
web page $\longrightarrow$ *assignment #11*

Trim in section 14.9 has a detailed proof of the Divergence Theorem. They try to interpret the meaning of

$$\oint\oint_S \vec{V}\cdot\hat{n}dS = \int\int\int_{\mathcal{V}}(\nabla\cdot\vec{V})d\mathcal{V}$$

This equation applies for any vector function $\vec{V}$, but is used most for velocity fields in fluids. When we consider $\vec{V}$, the theorem concerns net outflow to inflow (m^3/s) for a region in space (like a sphere).

The left side of the equation is a surface integral of V over a closed surface, S in 3-D space with $\hat{n}$ being the outward normal to each dS.

We recall that $\vec{V}\cdot\hat{n}dS$ gives the flow rate (m^3/s). When we add this up over the entire surface (as in the LHS of the equation) we obtain the **net flow rate crossing the closed surface**.

i.e.

$$\text{net outward flow}(m^3/s) - \text{net inward flow}(m^3/s)$$

The right hand side of the equation is a calculation of $\nabla\cdot\vec{V}$ for each differential volume, $d\mathcal{V}$ inside the surface S. We then add them all up.

For the differential volume, $\mathcal{V}$

$$\begin{aligned}
\text{inflow } (m^3/s) &= u(x)\cdot area + v(y)(\Delta x)(\Delta z)\\
\text{outflow } (m^3/s) &= u(x+\Delta x)\cdot(\Delta y)(\Delta z) + v(y+\Delta y)(\Delta x)(\Delta z)
\end{aligned}$$

The net flow is then

$$\text{outflow} - \text{inflow} = (\Delta x)(\Delta y)(\Delta z)\left[\frac{u(x+\Delta x)-u(x)}{\Delta x} + \frac{v(y+\Delta y)-v(y)}{\Delta y}\right]$$

$$
\begin{aligned}
&= (d\mathcal{V})\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
&= (\nabla \cdot \vec{V})d\mathcal{V}
\end{aligned}
$$

Therefore the RHS $(\nabla \cdot \vec{V})d\mathcal{V}$ gives the net outflow minus inflow for a volume element $d\mathcal{V}$.

The integral $\int\int\int_{\mathcal{V}}$, adds up the differential flow for all volume elements, $d\mathcal{V}$ inside of surface S. There is a cancellation of terms because the outflow from one $\Delta\mathcal{V}$ becomes the inflow to the next volume.

When we sum over all $\Delta\mathcal{V}$, we are left with the difference between the inflow and the outflow at the boundaries of the volume.

$$
\begin{aligned}
\oint\oint_S \vec{V} \cdot \vec{n} dS &= \int\int\int_{\mathcal{V}} (\nabla \cdot \vec{V})d\mathcal{V} \\
\text{outflow} - \text{inflow}(m^3/s) & \quad \text{triple sum of all outflow} - \text{inflow} \\
\text{across boundary surface} & \quad \text{for } \Delta\mathcal{V} \text{ volumes inside} \\
S \text{ of volume } \mathcal{V} \text{ in 3D space} & \quad \text{the volume } \mathcal{V} \text{ in 3D space}
\end{aligned}
$$

Example: 4.8

Given

$$\vec{V} = \hat{\imath}(1 + x) + \hat{\jmath}(1 + y^2) + \hat{k}(1 + z^3)$$

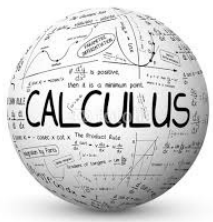
verify the divergence theorem for a cube, where $0 \leq x, y, z \leq 1$ i.e. show that

$$\oint\oint_S \vec{V} \cdot \hat{n} dS = \int\int\int_{\mathcal{V}} (\nabla \cdot \vec{V})d\mathcal{V}$$

where

$$
\begin{aligned}
S &= \text{cube surface (closed)} \\
\mathcal{V} &= \text{interior volume of the cube}
\end{aligned}
$$

Stoke's Theorem



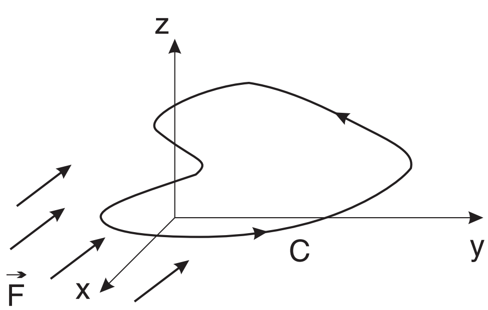
Reading
Trim 14.10 $\longrightarrow$ *Stoke's Theorem*

Assignment
web page $\longrightarrow$ *assignment #11*

The formal proof is offered in Trim 14.10.

$$\oint_C \vec{F} \cdot d\vec{r} = \int \int_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

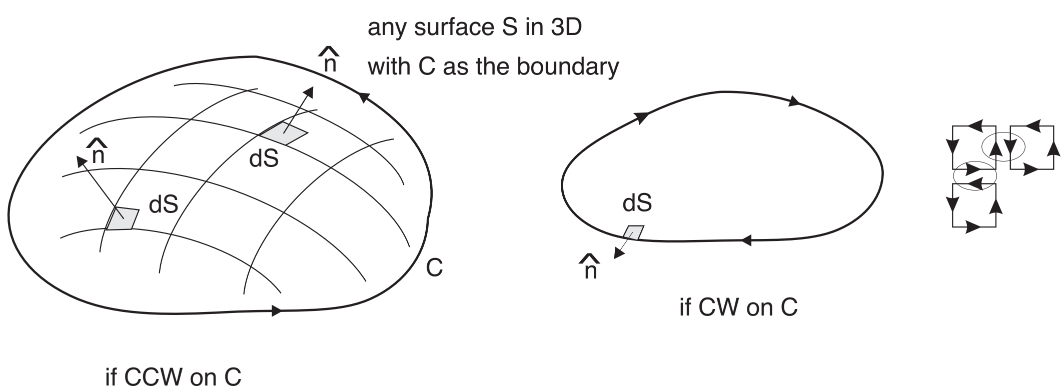
This has a similar meaning to Green's theorem but now in 3D space instead of a plane.



$\vec{F}(x, y, z)$ is a 3D force field. The LHS of the equations is the work done when the object moves once in a CCW direction along the path C in 3D.
The RHS is the work computed over a surface integral $\int \int_S$.

Like Greens theorem, it works because of the interior cancellations of work, when we move around a surface element dS inside C.

Note: the unit normal, $\hat{n}$ for the LHS is based on a right hand rule as follows.



The work on all internal surfaces cancel, leaving only the surface work in the CCW direction.

Example: 4.9

Given: $\vec{F} = \hat{i}x + \hat{j}2z + \hat{k}y$ (a force field in 3D).

The closed path C is given by the intersection of:

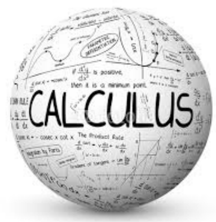
$$\begin{aligned} x^2 + y^2 &= 4 \\ z &= 4 - x - y \end{aligned}$$

The object moves once in a CW direction around C starting at $(2, 0, 2)$.

Verify Stoke' theorem:

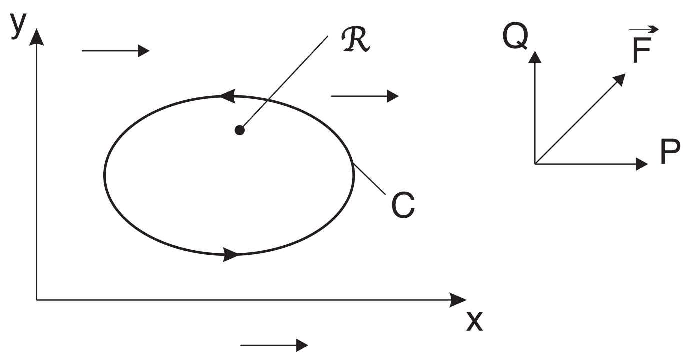
$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

Green's Theorem



Reading
Trim 14.6 ⟶ *Green's Theorem*

Assignment
web page ⟶ *assignment #10*



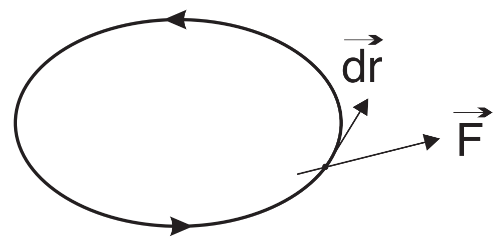
The theorem involves <u>work</u> done on an object by a 2D force field. The 2D force field is given by

$$\boldsymbol{F(x,y) = \hat{\imath}P(x,y) + \hat{\jmath}Q(x,y)}$$

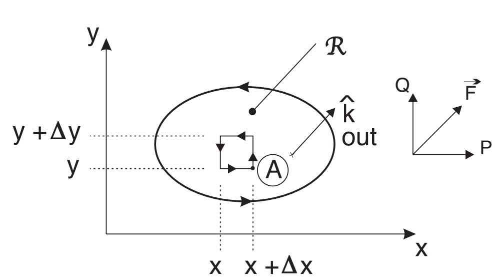
We will examine an object that moves once CCW around a loop in $\vec{F}$. The region inside the loop is defined as $\mathcal{R}$. The normal vector for the region $\mathcal{R}$ is $\hat{k}$ (outwards) to the right for a CCW motion of C. The theorem states

$$\oint_C P\,dx + Q\,dy = \int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

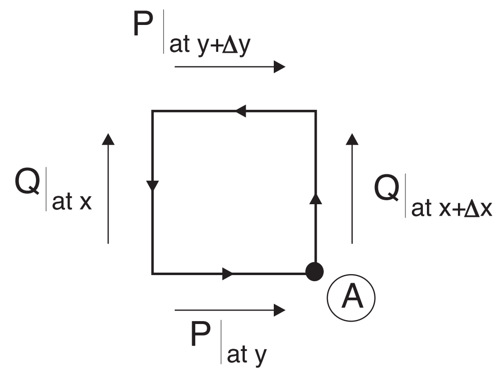
The **left hand side** of the equation, $\oint_C \vec{F}\cdot d\vec{r}$ in 2D is the work done by the field $\vec{F}$ on the object as it moves on C. This consists of force times distance for each $d\vec{r}$ added up over C.



The **right hand side** of the equation, $\int\int_{\mathcal{R}} (\nabla \times \vec{F})\cdot \hat{k}\,dx\,dy$ in 2D gives the direction of travel on C for the work given by the LHS. Maps movement of an element $\Delta x \Delta y$ as it moves around from A to A.



The force component times the distance for each side gives the work done.

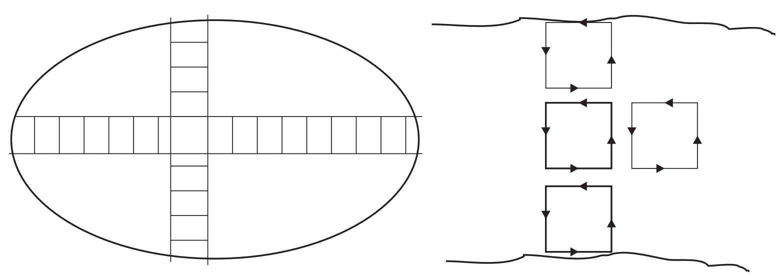


$$
\begin{aligned}
\boldsymbol{Work} &= \boldsymbol{Q}|_{\boldsymbol{at\ x+\Delta x}}\,\boldsymbol{\Delta y} - \boldsymbol{P}|_{\boldsymbol{at\ y+\Delta y}}\,\boldsymbol{\Delta x} - \boldsymbol{Q}|_{\boldsymbol{at\ x}}\,\boldsymbol{\Delta y} + \boldsymbol{P}|_{\boldsymbol{at\ y}}\,\boldsymbol{\Delta x} \\
&= (\boldsymbol{\Delta x \Delta y})\left[\frac{\boldsymbol{Q}|_{\boldsymbol{at\ x+\Delta x}} - \boldsymbol{Q}|_{\boldsymbol{at\ x}}}{\boldsymbol{\Delta x}} - \frac{\boldsymbol{P}|_{\boldsymbol{at\ y+\Delta y}} - \boldsymbol{P}|_{\boldsymbol{at\ y}}}{\boldsymbol{\Delta y}}\right]
\end{aligned}
$$

In the limit, the work done by $\vec{\boldsymbol{F}}$ to move the object CCW around $\boldsymbol{dxdy}$ area is

$$
\left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}}\right)\boldsymbol{dxdy}
$$

Now we can sum up for all $\boldsymbol{dxdy}$ elements inside $\boldsymbol{C}$.



There are some cancellations (+) (-) for all interior $\boldsymbol{\Delta x}, \boldsymbol{\Delta y}$ paths. Therefore when we do $\oint\oint_{\mathcal{R}}$, we are left with the work terms on the boundaries of $\mathcal{R}$.

$$
\oint_{\boldsymbol{C}} \boldsymbol{Pdx} + \boldsymbol{Qdy} = \int\int_{\mathcal{R}} \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}}\right)\boldsymbol{dxdy}
$$

work when we	**sum of all work if**
move once CCW around	**move once CCW around all**
boundary curve $\boldsymbol{C}$	**$(\boldsymbol{dxdy})$ area elements of $\mathcal{R}$**
	inside $\boldsymbol{C}$

Example: 4.10

Given a 2D force field, $\vec{F}(x,y) = \hat{i}(xy^3) + \hat{j}(x^2y)$ and a path C in the field:

Verify Green's theorem

$$\oint_C P dx + Q dy = \int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

with

$$\begin{aligned} P &= xy^3 \\ Q &= x^2y \end{aligned}$$