Laplace's equation in the Polar Coordinate System

As I mentioned in my lecture, if you want to solve a partial differential equation (PDE) on the domain whose shape is a 2D disk, it is much more convenient to represent the solution in terms of the polar coordinate system than in terms of the usual Cartesian coordinate system. For example, the behavior of the drum surface when you hit it by a stick would be best described by the solution of the wave equation in the polar coordinate system. In this note, I would like to derive Laplace's equation in the polar coordinate system in details.

Recall that **Laplace's equation** in $\mathbb{R}^2$ in terms of the usual (i.e., **Cartesian**) (x, y) coordinate system is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0. \tag{1}$$

The Cartesian coordinates can be represented by the polar coordinates as follows:

$$\begin{cases} x = r\cos\theta; \\ y = r\sin\theta. \end{cases} \tag{2}$$

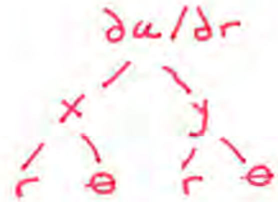
Let us first compute the partial derivatives of x, y w.r.t. r, θ:

$$\begin{cases} \dfrac{\partial x}{\partial r} = \cos\theta, & \dfrac{\partial x}{\partial \theta} = -r\sin\theta; \\ \dfrac{\partial y}{\partial r} = \sin\theta, & \dfrac{\partial y}{\partial \theta} = r\cos\theta. \end{cases} \tag{3}$$

To do so, let's compute $\dfrac{\partial u}{\partial r}$ first. We will use the **Chain Rule** since (x, y) are functions of (r, θ) as shown in (2).

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \text{using (3)} \\ &= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}. \end{aligned} \tag{4}$$

Now, let's compute $\dfrac{\partial^2 u}{\partial r^2}$. Noticing that both $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are functions of (x, y)



no direct path to r

and using (3), we have

$$
\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= \cos\theta \frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin\theta \frac{\partial}{\partial r}\frac{\partial u}{\partial y} \\
&= \cos\theta \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial r} \right) + \sin\theta \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \right) \\
&= \cos^2\theta \frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 u}{\partial y^2}. \qquad (5)
\end{aligned}
$$

Similarly, let's compute $\frac{\partial u}{\partial \theta}$ and $\frac{\partial^2 u}{\partial \theta^2}$.

$$
\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\
&= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \\
&= -r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}.
\end{aligned}
$$

$a \cdot b + c \cdot d \rightarrow$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta \frac{\partial}{\partial \theta}\frac{\partial u}{\partial x} - r\sin\theta \frac{\partial u}{\partial y} + r\cos\theta \frac{\partial}{\partial \theta}\frac{\partial u}{\partial y} \\
&= -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \theta} \right) - r\sin\theta \frac{\partial u}{\partial y} + r\cos\theta \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \right) \\
&= -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta \left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial x \partial y} r\cos\theta \right) \\
&\quad - r\sin\theta \frac{\partial u}{\partial y} + r\cos\theta \left(\frac{\partial^2 u}{\partial x \partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2} r\cos\theta \right) \\
&= -r\left(\cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y} \right) + r^2 \left(\sin^2\theta \frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta \frac{\partial^2 u}{\partial y^2} \right)
\end{aligned}
$$

Dividing both sides by r^2 and using (4), we have

$$
\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta \frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta \frac{\partial^2 u}{\partial y^2} \qquad (6)
$$

Finally, adding (5) and (6), using the obvious relation $\cos^2\theta + \sin^2\theta = 1$, we have

$$
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},
$$

OR → use product rule

$$
a\frac{\partial b}{\partial \theta} + \underbrace{b\frac{\partial a}{\partial \theta}} + c\frac{\partial d}{\partial \theta} + \underbrace{d\frac{\partial c}{\partial \theta}}
$$

direct path to θ

which can be cleaned up as:

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.}$$

Hence, Laplace's equation (1) becomes:

$$\boxed{u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.}$$

Once we derive Laplace's equation in the polar coordinate system, it is easy to represent the heat and wave equations in the polar coordinate system. For the heat equation, the solution $u(x, y, t) = u(r, \theta, t)$ satisfies

$$\boxed{u_t = k(u_{xx} + u_{yy}) = k\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad k > 0\text{: diffusivity,}}$$

whereas for the wave equation, we have

$$\boxed{u_{tt} = c^2(u_{xx} + u_{yy}) = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \quad c > 0\text{: wave velocity.}}$$