we are left with the work terms on the boundaries of $\mathcal{R}$.

$$\oint_C Pdx + Qdy \quad = \quad \int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

work when we	sum of all work if
move once CCW around	move once CCW around all
boundary curve C	$(dxdy)$ area elements of $\mathcal{R}$
	inside C

Example: 4.10

Given a 2D force field, $\vec{F}(x,y) = \hat{\imath}(xy^3) + \hat{\jmath}(x^2y)$ and a rectangular path C defined as $0 \leq x \leq 1$ and $-1 \leq y \leq 1$ where we move around the path in a counter clockwise direction:

Verify Green's theorem

$$\oint_C Pdx + Qdy = \int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

with

$$P = xy^3$$
$$Q = x^2y$$

LHS: For the general curve C, we need the parametric form to compute $\oint_C Pdx+Qdy$. However, because of the shape of C in this example, a parametric representation of the curve is not required.

1. on C_1:

$$y = 1 \qquad dy = 0 \qquad P = x \qquad Q = x^2$$

$$W = \int_{C_1} Pdx + Qdy = \int_{x=1}^{0} xdx = \left. \frac{x^2}{2} \right|_1^0 = -\frac{1}{2}$$

2. on C_2:

$$x = 0 \qquad dx = 0 \qquad P = 0 \qquad Q = 0$$

$$\int_{C_2} Pdx + Qdy = 0$$

3. on C_3:

$$y = -1 \qquad dy = 0 \qquad P = -x \qquad Q = -x^2$$

$$\int_{C_3} Pdx + Qdy = \int_{x=0}^{1} -xdx = -\frac{x^2}{2}\Big|_0^1 = -\frac{1}{2}$$

4. on C_4:

$$x = 1 \qquad dx = 0 \qquad P = y^3 \qquad Q = y$$

$$\int_{C_4} Pdx + Qdy = \int_{y=-1}^{1} ydy = \frac{y^2}{2}\Big|_{-1}^1 = 0$$

Therefore

$$W = \oint_C Pdx + Qdy = -\frac{1}{2} - \frac{1}{2} = -1\ Joules$$

A negative value of W implies that work has to be supplied by the object.

RHS:

$$\int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int\int_{\mathcal{R}} \left(2xy - 3xy^2\right) dxdy$$

In this particular case the calculation of the double integral is made easier because $\mathcal{R}$ limits are all constants.

$$\begin{aligned} \int_{x=0}^{1} \left[\int_{y=-1}^{1} (2xy - 3xy^2)dy\right] dx &= \int_{x=0}^{1} \left[(xy^2 - xy^3)\Big|_{y=-1}^{1}\right] dx \\ &= \int_{x=0}^{1} (-2x)dx = -x^2\Big|_0^1 = -1\ Joules \end{aligned}$$

Therefore LHS = RHS.

We can use Green's theorem to change a line integral computation of W to a $\int\int_{\mathcal{R}}$ instead.